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Complete intersections in affine monomial curves

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Abstract

Let P be the toric ideal of an affine monomial curve over an arbitrary field. Using a combinatorial-geometric approach, we characterize when P is a complete intersection in terms of certain arithmetical conditions on binary trees.

1 Introduction

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Given a subset I of R we denote its zero set in $\mathbb{A}_k^n$ by $V(I)$ and given a subset $X \subset \mathbb{A}_k^n$ we denote its vanishing ideal in R by $I(X)$. As usual we use x^a as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. A *binomial* in R is a difference of two monomials, that is $f = x^a - x^b$ for some $a, b \in \mathbb{N}^n$. An ideal of R generated by binomials is called a *binomial ideal.*

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Let $\underline{d} = \{d_1, \ldots, d_n\}$ be a set of distinct positive integers and consider the monomial curve

$$\Gamma = \{(t^{d_1}, \ldots, t^{d_n}) \in \mathbb{A}^n_k \,|\, t \in k\}\,.$$

The homomorphism of k-algebras:

$$\phi: R \to k[t]; \qquad x_i \longmapsto t^{d_i}$$

is graded if we set $\deg(x_i) = d_i$ and $\deg(t) = 1$. The image of ϕ will be denoted by $k[\Gamma]$ and its kernel will be denoted by P. The ideal P is called the *toric ideal* of Γ. Since $k[t]$ is integral over $k[\Gamma]$ we have $\mathrm{ht}(P) = n - 1$. By [13, Proposition 7.1.2], the toric ideal P is generated by binomials. According to [6, Lemma 3.4], if $\gcd(\underline{d}) = 1$, Γ is an affine toric variety, that is $\Gamma = V(P)$. If k is an infinite field, we get $I(\Gamma) = P$, see [13, Corollary 7.1.12]. Note that the ideal $P \subset R$ is *quasi-homogeneous*, i.e., homogeneous if one gives degree d_i to variable x_i, and one says that the *degree* of a quasi-homogeneous binomial $x^a - x^b$ in P is $a_1d_1 + \cdots + a_nd_n$.

The prime ideal P is called a *binomial set theoretic complete intersection* if there exists a system of binomials $g_1, \ldots, g_{n-1}$ such that $P = \mathrm{rad}(g_1, \ldots, g_{n-1})$. If $P = (g_1, \ldots, g_{n-1})$ we call P a *complete intersection.* In [4] it is shown that P is generated up to radical by n binomials. In positive characteristic, P is always a binomial set theoretic complete intersection (see [10]). A clever constructive proof of this result, using diophantine equations and linear algebra, can be found in [1]. If k is of characteristic zero, P is a binomial set theoretic complete intersection if and only if it is a complete intersection by [2, Theorem 4]. As a byproduct, we will recover this result in Section 2 (Corollary 2.6).

There is a description of complete intersection semigroups of $\mathbb{N}$ given in [3], see also [7] for a generalization of this description to semigroups of arbitrary dimension. In the area of complete intersection toric ideals there are some recent papers, see the introduction of [11] and the references there. We present a combinatorial-geometric approach that leads to a new effective criterion for complete intersection toric ideals of affine monomial curves. This approach is different in nature to that of [3]. Using the notion of binary tree we are able to uncover a combinatorial-arithmetical structure of complete intersections. A binary tree representing a complete intersection will contain essential information of the curve Γ and its semigroup $\mathbb{N}\underline{d}$, for instance the defining equations of $k[\Gamma]$ and the Frobenius number of the numerical semigroup $\mathbb{N}\underline{d}$ (Remark 4.5).

The contents of this paper are as follows. In Section 2, we first claim that any primary binomial ideal over a field of characteritic zero is radical (Proposition 2.3). Its proof uses ideas introduced by Shalom Eliahou [4, 5]. Next, using a result of [6] we observe (Proposition 2.5) that P is a complete intersection if and only if there are binomials $g_1, \ldots, g_{n-1}$ in P with $g_i = x^{\alpha_i} - x^{\beta_i}$ such that

(a) $\ker(\psi) = \mathbb{Z}\widehat{g}_1 + \cdots + \mathbb{Z}\widehat{g}_{n-1}$, where $\widehat{g}_i = \alpha_i - \beta_i$ and ψ is the linear map $\psi\colon \mathbb{Z}^n \to \mathbb{Z}$ induced by $\psi(e_i) = d_i$,

(b) $V(g_1, \ldots, g_{n-1}, x_i) = \{0\}$ for $i = 1, \ldots, n$.

For arbitrary binomials, we express the geometric condition (b) in purely combinatorial terms using the notion of "binary tree labeled by $\{1, \ldots, n\}$ and compatible with $g_1, \ldots, g_{n-1}$" (Theorem 3.7). This result is interesting in its own right because it links geometry with discrete mathematics (digraphs) and because it can be used for arbitrary binomial ideals that need not be toric. Next, assuming that (b) holds, we characterize condition (a) in terms of arithmetical conditions on the d_i's (Proposition 4.2). Putting it all together, we present a combinatorial-arithmetical structure theorem that characterizes when P is a complete intersection (Theorem 4.3).

2 Binomial ideals and their radicals

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Throughout this section, I will denote a binomial ideal of R generated by $\{g_1, \ldots, g_r\}$, where $g_i = x^{\alpha_i} - x^{\beta_i}$ for $i = 1, \ldots, r$. Note that a binomial ideal does not contain any monomial of R. We denote by $\mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\}$ the subgroup of $\mathbb{Z}^n$ generated by $\widehat{g}_1 = \alpha_1 - \beta_1, \ldots, \widehat{g}_r = \alpha_r - \beta_r$. Since $\mathrm{rad}(I)$ is again a binomial ideal (see [8, Theorem 9.4 and Corollary 9.12]), $\mathrm{rad}(I)$ is generated by $\{h_1, \ldots, h_s\}$ where $h_i = x^{\gamma_i} - x^{\delta_i}$ for $i = 1, \ldots, s$. If I is primary, then $h_1, \ldots, h_s$ can be chosen such that x^{γ_i} and x^{δ_i} have no common variables.

Let G be a subgroup of $\mathbb{Z}^n$. Following [4], we define an equivalence relation $\sim_G$ on the set of monomials of R by $x^\alpha \sim_G x^\beta$ if and only if $\alpha - \beta \in G$. This relation is compatible with the product. A non zero polynomial $f = \sum_\alpha \lambda_\alpha x^\alpha$ is *simple* with respect to $\sim_G$ if all its monomials with non zero coefficient are equivalent under $\sim_G$. An arbitrary non zero polynomial f in R is uniquely expressed as the sum of simple polynomials that we call its *simple components with respect to* G: $f = f_1 + \cdots + f_m$ such that f_i is simple and if $i \neq j$ and x^α, x^β are monomials in f_i and f_j respectively, then $x^\alpha \not\sim_G x^\beta$.

For convenience we recall the following result about the behaviour of simple components valid in any characteristic.

Lemma 2.1 ([6, Lemma 2.2]) *Given a non zero polynomial f in R, if $f \in I$ then any simple component of f with respect to $\mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\}$ belongs to I.*

Lemma 2.2 *If the characteristic of k is zero, then $\mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\} = \mathbb{Z}\{\widehat{h}_1, \ldots, \widehat{h}_s\}$.*

Proof. Set $G_1 = \mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\}$ and $G_2 = \mathbb{Z}\{\widehat{h}_1, \ldots, \widehat{h}_s\}$. Since $g_i \in \mathrm{rad}(I)$, then by Lemma 2.1, any simple component of g_i with respect to G_2 belongs to $\mathrm{rad}(I)$. Therefore, $\alpha_i \sim_{G_2} \beta_i$ otherwise $\mathrm{rad}(I)$ would contain x^{α_i}, which is impossible. This proves that $G_1 \subset G_2$. Observe that this holds in any characteristic.

To show the reverse containment, we adapt the argument given in the proof of [6, Proposition 2.4]. Since $h_i = x^{\gamma_i} - x^{\delta_i} \in \mathrm{rad}(I)$, then $h_i^{p^m} \in I$ for $m \gg 0$ and p an arbitrary prime number. We claim that $x^{p^m\gamma_i} \sim_{G_1} x^{p^m\delta_i}$. Consider the equality

$$h_i^{p^m} = \sum_{s=0}^{p^m} (-1)^s \binom{p^m}{s} (x^{\gamma_i})^{p^m-s}(x^{\delta_i})^s .$$

If $x^{p^m\gamma_i}$ and $x^{p^m\delta_i}$ are not in the same simple component of $h_i^{p^m}$ with respect to G_1, then there is a non empty subset $S \subset \{1, \ldots, p^m-1\}$ such that the polynomial

$$f = x^{p^m\gamma_i} + \sum_{s\in S} (-1)^s \binom{p^m}{s} (x^{\gamma_i})^{p^m-s}(x^{\delta_i})^s$$

is a simple component of $h_i^{p^m}$ with respect to G_1. By Lemma 2.1, $f \in I$, and hence

$$f(1, \ldots, 1) = 0 = 1 + \sum_{s\in S} (-1)^s \binom{p^m}{s},$$

a contradiction if the characteristic of k is zero because $\binom{p^m}{s} \equiv 0 \mod(p)$ for $1 \leq s \leq p^m - 1$. Therefore, $x^{p^m\gamma_i} \sim_{G_1} x^{p^m\delta_i}$, and consequently $p^m(\gamma_i - \delta_i) \in G_1$. If we pick another prime number $q \neq p$ and $t \gg 0$, repeating the previous argument, we obtain $q^t(\gamma_i - \delta_i) \in G_1$, and hence $\gamma_i - \delta_i \in G_1$, as required. □

Proposition 2.3 *Assume that the characteristic of k is zero. If I is primary, then* $\mathrm{rad}(I) = I$.

Proof. Let us show that $h_i = x^{\gamma_i} - x^{\delta_i}$ belongs to I for all $i = 1, \ldots, s$. By Lemma 2.2, we can write

$$\gamma_i - \delta_i = \eta_1(\alpha_1 - \beta_1) + \cdots + \eta_r(\alpha_r - \beta_r) \qquad (\eta_i \in \mathbb{Z}) .$$

By substituting $-g_i$ for g_i if necessary, we may assume that $\eta_1, \ldots, \eta_r \in \mathbb{N}$. Expanding the right hand side of the equality

$$\frac{h_i}{x^{\delta_i}} = \left[\left(\frac{x^{\alpha_1}}{x^{\beta_1}} - 1\right) + 1\right]^{\eta_1} \cdots \left[\left(\frac{x^{\alpha_r}}{x^{\beta_r}} - 1\right) + 1\right]^{\eta_r} - 1$$

readily gives a monomial x^γ such that $x^\gamma h_i \in I$. If $h_i \notin I$, then $(x^\gamma)^\ell \in I$ for some $\ell \geq 1$ because I is primary, but this is impossible. Thus $h_i \in I$, as required. □

Remark 2.4 Note that Proposition 2.3 fails if the characteristic of the field k is positive. For example, the primary ideal $I = (x^{10} - y^{15}) \subset \mathbb{F}_5[x, y]$ is not radical. In this example, $\mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\} \neq \mathbb{Z}\{\widehat{h}_1, \ldots, \widehat{h}_s\}$. However, one obtains as a direct consequence of the proof of Proposition 2.3 that if $\mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\} = \mathbb{Z}\{\widehat{h}_1, \ldots, \widehat{h}_s\}$ and I is primary, then $I = \mathrm{rad}(I)$. This observation is useful in the proof of our next result, which is one of the keys to our main result (Theorem 4.3).

Proposition 2.5 *Let k be an arbitrary field and let $\mathcal{B} = \{g_1, \ldots, g_{n-1}\}$ be a set of binomials in P, the toric ideal of the monomial curve Γ. Then $P = (\mathcal{B})$ if and only if*

(a) $\ker(\psi) = \mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_{n-1}\}$ *and*

(b) $V(g_1, \ldots, g_{n-1}, x_i) = \{0\}$ *for* $i = 1, \ldots, n$.

Proof. If $P = (\mathcal{B})$ then (a) follows at once from [6, Proposition 2.3], and (b) follows from [6, Theorem 3.1(b)]. Conversely, if (a) and (b) hold then by [6, Theorem 3.1], one has $\mathrm{rad}(\mathcal{B}) = P$. Let $\{h_1, \ldots, h_s\}$ be a set of generators of P consisting of binomials. Notice that $\ker(\psi) = \mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_r\}$ by (a), and $\ker(\psi) = \mathbb{Z}\{\widehat{h}_1, \ldots, \widehat{h}_s\}$ by [6, Proposition 2.3]. Thus, since $(\mathcal{B})$ is a complete intersection and its radical is a prime ideal, $(\mathcal{B})$ is radical by Remark 2.4, and hence $P = (\mathcal{B})$. □

We end this section recovering a result that holds for toric ideals of arbitrary dimension over a field of characteristic zero, see also [11, Corollary 3.10] for a recent generalization.

Corollary 2.6 ([2, Theorem 4]) *Let $\mathfrak{p}$ be a toric ideal of R. If $\mathfrak{p}$ is a binomial set theoretic complete intersection, then $\mathfrak{p}$ is a complete intersection.*

Proof. Set $r = \dim R/\mathfrak{p}$. By hypothesis, there are $g_1, \ldots, g_{n-r}$ binomials of R such that $\mathrm{rad}(g_1, \ldots, g_{n-r}) = \mathfrak{p}$. Since the ideal $(g_1, \ldots, g_{n-r})$ is primary because it is a complete intersection and its radical is a prime ideal, the result follows from Proposition 2.3. □

3 Binary trees in binomial ideals

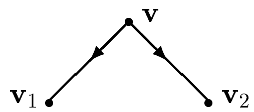
Definition 3.1 A *binary tree* is a connected directed rooted tree such that: (i) two edges leave the root and every other vertex has either degree 1 or 3, (ii) if a vertex has degree 3, then one edge enters the vertex and the other two edges leave the vertex, and (iii) if a vertex has degree 1, then one edge enters the vertex. The vertices of degree 1 are called *terminal*. For convenience we regard an isolated vertex as a binary tree.

Lemma 3.2 *If G is a binary tree with n terminal vertices, then the number of non-terminal vertices of G is $n-1$.*

Proof. It follows by induction on n. $\square$

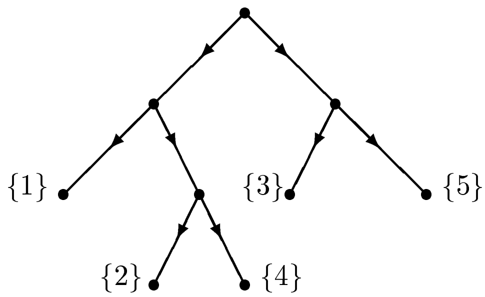
Definition 3.3 A binary tree G is said to be *labeled by* $[\![1,n]\!] := \{1,\ldots,n\}$ if its terminal vertices are labeled by $\{1\}$, …, $\{n\}$. Extending this definition, we will also consider binary trees with n terminal vertices labeled by arbitrary finite subsets of $\mathbb{N}$ with n elements.

If G is a binary tree labeled by $[\![1,n]\!]$ and $\mathbf{v}$ is a non-terminal vertex of G, consider $\mathbf{v}_1$ and $\mathbf{v}_2$, the two vertices of G such that



is a subgraph of G, and denote by G_1, resp. G_2, the subtree of G whose root is $\mathbf{v}_1$, resp. $\mathbf{v}_2$. We denote by $\ell_1[\mathbf{v}]$ and $\ell_2[\mathbf{v}]$ the two disjoint subsets of $[\![1,n]\!]$ formed by the union of labels of the terminal vertices of G_1 and G_2 respectively.

Example 3.4 The following binary tree is labeled by $[\![1,5]\!]$:



and if $\mathbf{v}$ is the root of G, then $\ell_1[\mathbf{v}] = \{1,2,4\}$ and $\ell_2[\mathbf{v}] = \{3,5\}$.

The *support* of a monomial x^a (resp. binomial $g = x^a - x^b$) is denoted by $\text{supp}(x^a) = \{i \mid a_i > 0\}$ (resp. $\text{supp}(g) = \text{supp}(x^a) \cup \text{supp}(x^b)$).

Definition 3.5 Let $\mathcal{B} = \{g_1,\ldots,g_{n-1}\}$ be a set of binomials of R with $g_i = x^{\alpha_i} - x^{\beta_i}$, $\text{supp}(x^{\alpha_i}) \cap \text{supp}(x^{\beta_i}) = \emptyset$, and $\alpha_i \neq 0$, $\beta_i \neq 0$ for all $i = 1,\ldots,n-1$, and let G be a binary tree labeled by $[\![1,n]\!]$. We say that G is *compatible* with $\mathcal{B}$ if, denoting by $\mathcal{F}$ the set of non-terminal vertices of G, there is a bijection

$$\mathcal{B} \xrightarrow{f} \mathcal{F}$$

such that $\text{supp}(x^{\alpha_i}) \subset \ell_1[f(g_i)]$ and $\text{supp}(x^{\beta_i}) \subset \ell_2[f(g_i)]$ for all $i \in \{1,\ldots,n-1\}$.

Example 3.6 The binary tree G labeled by $[\![1,5]\!]$ in Example 3.4 is compatible with the set of binomials

$$\{g_1 = x_1^2 x_2^4 - x_3 x_5\,,\ g_2 = x_1 - x_2 x_4\,,\ g_3 = x_3^4 - x_5^2\,,\ g_4 = x_2 - x_4^7\}.$$

The next result will be used later to prove our main result. It is interesting in its own right because it characterizes a geometric condition "$V(\mathcal{B}, x_i) = \{0\}$" that occurs in the study of toric curves (see [4, 6]) in terms of a combinatorial notion "labeled binary tree". In addition this result holds for arbitrary binomials not necessarily inside of a toric ideal.

Theorem 3.7 *Let $\mathcal{B} = \{g_1, \ldots, g_{n-1}\}$ be a set of binomials of R such that $g_i = x^{\alpha_i} - x^{\beta_i}$, $\mathrm{supp}(x^{\alpha_i}) \cap \mathrm{supp}(x^{\beta_i}) = \emptyset$, and $\alpha_i \neq 0$, $\beta_i \neq 0$ for all $i = 1, \ldots, n-1$. Then the following two conditions are equivalent:*

(1) $V(\mathcal{B}, x_i) = \{0\}$ *for all* $i = 1, \ldots, n$.

(2) *There exists a binary tree G labeled by $[\![1,n]\!]$ which is compatible with $\mathcal{B}$.*

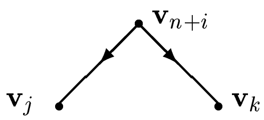
Proof. (1) $\Rightarrow$ (2): Set $V_1 := \{1\}$, $\ldots$, $V_n := \{n\}$ and consider the partition $\mathcal{F}_1 := \{V_1, \ldots, V_n\}$ of $[\![1,n]\!]$. Let us show that there exist $V_{n+1}, \ldots, V_{2n-1}$, subsets of $[\![1,n]\!]$, and $\mathcal{F}_2, \ldots, \mathcal{F}_n$, partitions of $[\![1,n]\!]$, such that, reindexing $g_1, \ldots, g_{n-1}$ if necessary, the following assertions hold for all $i \in \{1, \ldots, n-1\}$:

(a) $V_{n+i} = V_j \cup V_k$ for some $V_j, V_k \in \mathcal{F}_i$, $j \neq k$.

(b) $\mathrm{supp}(x^{\alpha_i}) \subset V_j$ and $\mathrm{supp}(x^{\beta_i}) \subset V_k$.

(c) $\mathcal{F}_{i+1} = (\mathcal{F}_i \setminus \{V_j, V_k\}) \cup \{V_{n+i}\}$.

Then, if we consider the digraph G with $2n-1$ vertices, denoted by $\mathbf{v}_1, \ldots, \mathbf{v}_{2n-1}$, where we connect $\mathbf{v}_{n+i}$ with $\mathbf{v}_j$ and $\mathbf{v}_k$ as follows:



whenever $V_{n+i} = V_j \cup V_k$ in (a), it is not hard to see that G is a binary tree labeled by $[\![1,n]\!]$. The root of G is $\mathbf{v}_{2n-1}$, and the set of its non-terminal vertices is $\mathcal{F} := \{\mathbf{v}_{n+1}, \ldots, \mathbf{v}_{2n-1}\}$. Moreover, by construction, for all $i \in \{1, \ldots, n-1\}$, one has that $\ell_1[\mathbf{v}_{n+i}] = V_j$ and $\ell_2[\mathbf{v}_{n+i}] = V_k$ for V_j and V_k in (a). Hence, by (b), G is compatible with $\mathcal{B}$ via the map $f : \mathcal{B} \to \mathcal{F}$, $g_i \mapsto \mathbf{v}_{n+i}$, and (2) will follow.

Let us first construct V_{n+1} and $\mathcal{F}_2$ satisfying (a), (b) and (c). We first claim that for all $i \in [\![1,n]\!]$, there exists an element $g_j \in \mathcal{B}$ such that either $\mathrm{supp}(x^{\alpha_j}) \subset V_i$ or $\mathrm{supp}(x^{\beta_j}) \subset V_i$ because otherwise, we have that the *ith* unit vector e_i of

$\mathbb{A}_k^n$ belongs to $V(\mathcal{B}, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ which is $\{0\}$ by (1). Since $|\mathcal{F}_1| = n$ and $|\mathcal{B}| = n-1$, by the pigeonhole principle there exists an element in $\mathcal{B}$, say g_1, and $V_j, V_k \in \mathcal{F}_1$ with $j \neq k$, such that $\text{supp}(x^{\alpha_1}) \subset V_j$ and $\text{supp}(x^{\beta_1}) \subset V_k$. Setting $V_{n+1} := V_j \cup V_k$ and $\mathcal{F}_2 := (\mathcal{F}_1 \setminus \{V_j, V_k\}) \cup \{V_{n+1}\}$, the statements (a), (b) and (c) hold for $i = 1$.

Assume now that for $i \in \{2, \dots, n-1\}$, we have constructed $V_{n+1}, \dots, V_{n+i-1}$ and $\mathcal{F}_2, \dots \mathcal{F}_i$ such that (a), (b) and (c) hold, and let us construct V_{n+i} and $\mathcal{F}_{i+1}$ satisfying (a), (b) and (c).

Observe first that for all $j \leq i-1$, $\text{supp}(g_j)$ is contained in some element of $\mathcal{F}_i$. Set $\mathcal{B}_i := \mathcal{B} \setminus \{g_1, \dots, g_{i-1}\}$. We claim that for each $V_k \in \mathcal{F}_i$, there exists $g_j \in \mathcal{B}_i$ such that either $\text{supp}(x^{\alpha_j}) \subset V_k$ or $\text{supp}(x^{\beta_j}) \subset V_k$. In order to prove this, we show that if there exists an element in $\mathcal{F}_i$, say $V_s = \{i_1, \dots, i_m\}$, that does not satisfy the claim, then $\alpha := e_{i_1} + \cdots + e_{i_m}$ belongs to $V(\mathcal{B})$, which is a contradiction by (1). Take $g_j \in \mathcal{B}$. If $g_j \in \mathcal{B}_i$, then $\text{supp}(x^{\alpha_j}) \not\subset V_s$ and $\text{supp}(x^{\beta_j}) \not\subset V_s$ by definition of V_s, and hence $g_j(\alpha) = 0$. If $g_j \notin \mathcal{B}_i$, i.e., if $j \leq i-1$, then $\text{supp}(g_j)$ is contained in some element of $\mathcal{F}_i$, say V_t. If $t = s$, i.e., if $\text{supp}(g_j) \subset V_s$, then $g_j(\alpha) = 1 - 1 = 0$. Otherwise, since $\mathcal{F}_i$ is a partition of $[\![1, n]\!]$ and $V_s, V_t \in \mathcal{F}_i$, one has that $V_s \cap V_t = \emptyset$, and hence $\text{supp}(g_j) \cap V_s = \emptyset$. Thus, $g_j(\alpha) = 0$, and the claim is proved.

We have proved that for each $V_k \in \mathcal{F}_i$, there exists $g_j \in \mathcal{B}_i$ such that either $\text{supp}(x^{\alpha_j}) \subset V_k$ or $\text{supp}(x^{\beta_j}) \subset V_k$. Since $|\mathcal{F}_i| = n - i + 1$ and $|\mathcal{B}_i| = n - i$, and using that $\mathcal{F}_i$ is a partition of $[\![1, n]\!]$, we get by the pigeonhole principle that there exist an element in $\mathcal{B}_i$, say g_i, and $V_j, V_k \in \mathcal{F}_i$ such that $\text{supp}(x^{\alpha_i}) \subset V_j$ and $\text{supp}(x^{\beta_i}) \subset V_k$. Setting $V_{n+i} := V_j \cup V_k$ and $\mathcal{F}_{i+1} = (\mathcal{F}_i \setminus \{V_j, V_k\}) \cup \{V_{n+i}\}$, the statements (a), (b) and (c) hold, and we are done.

(2) $\Rightarrow$ (1): The proof is by induction on n, the number of variables. The result is clear if $n = 2$. Denoting by $\mathbf{v}$ the root of G, we may assume without loss of generality, that $\ell_1[\mathbf{v}] = [\![1, r]\!]$ and $\ell_2[\mathbf{v}] = [\![r+1, n]\!]$ for some $r \in \{1, \dots, n-1\}$. Then, if G_1 and G_2 are the two connected components of the digraph obtained from G by removing the vertex $\mathbf{v}$ and the two edges leaving $\mathbf{v}$, one has that G_1 and G_2 are binary trees labeled by $[\![1, r]\!]$ and $[\![r+1, n]\!]$ respectively. Reindexing the g_i's if necessary, we may also assume that G_1 is compatible with $\mathcal{B}_1 := \{g_2, \dots, g_r\}$, G_2 is compatible with $\mathcal{B}_2 := \{g_{r+1}, \dots, g_{n-1}\}$, and $g_1 = x^{\alpha_1} - x^{\beta_1}$ with $\text{supp}(x^{\alpha_1}) \subset [\![1, r]\!]$ and $\text{supp}(x^{\beta_1}) \subset [\![r+1, n]\!]$. Then, $\text{supp}(g_i) \subset [\![1, r]\!]$ if $i = 2, \dots, r$, and $\text{supp}(g_i) \subset [\![r+1, n]\!]$ if $i = r+1, \dots, n-1$. Moreover, applying the induction hypothesis, one has that $V(\mathcal{B}_1, x_i) = \{0\}$ for all $i = 1, \dots, r$, and $V(\mathcal{B}_2, x_i) = \{0\}$ for all $i = r+1, \dots, n$. Fix $i \in [\![1, n]\!]$ and take $a \in V(\mathcal{B}, x_i)$. The result will be proved if we show that $a = 0$. By symmetry, we may assume that $1 \leq i \leq r$. The vector $a = (a_1, \dots, a_n)$ can be decomposed as $a = b + c$, where $b = (a_1, \dots, a_r, 0, \dots, 0)$. Then $b \in V(\mathcal{B}_1, x_i)$, and hence $b = 0$. On the

other hand, $g_1(a) = 0$ implies that $a_j = 0$ for some $j \in \{r+1, \dots, n\}$. Thus $c \in V(\mathcal{B}_2, x_j)$ which is $\{0\}$, and hence $a = 0$, as required. □

4 Complete intersections

Let $\underline{d} = \{d_1, \dots, d_n\}$ be a set of distinct positive integers, and consider the monomial curve $\Gamma \subset \mathbb{A}_k^n$ and the toric ideal $P \subset k[x_1, \dots, x_n]$ defined in the introduction. The exact sequence

$$0 \longrightarrow \ker(\psi) \longrightarrow \mathbb{Z}^n \xrightarrow{\psi} \mathbb{Z} \longrightarrow 0; \quad e_i \stackrel{\psi}{\longmapsto} d_i$$

is related to P as follows. If $g = x^a - x^b$ is a binomial, then $g \in P$ if and only if $a - b \in \ker(\psi)$.

Given a binomial $g = x^a - x^b$, we set $\widehat{g} = a - b$. If $\alpha = (\alpha_i) \in \mathbb{Z}^n$, its support is given by $\mathrm{supp}(\alpha) = \{i | \alpha_i \neq 0\}$. Any $\alpha \in \mathbb{Z}^n$ can be written as $\alpha = \alpha^+ - \alpha^-$, where α^+ and α^- are vectors in $\mathbb{N}^n$ with disjoint support. If $S \subset \mathbb{N}^n$, the subsemigroup (resp. subgroup) of $\mathbb{N}^n$ (resp. $\mathbb{Z}^n$) generated by S will be denoted by $\mathbb{N}S$ (resp. $\mathbb{Z}S$).

Definition 4.1 Let G be a binary tree labeled by $[\![1, n]\!]$, and consider a set of vectors in $\mathbb{Z}^n$, $W = \{w_1, \dots, w_{n-1}\}$. We say that G is *compatible* with W if G is compatible with the set of binomials $\{x^{w_i^+} - x^{w_i^-};\ i = 1, \dots, n-1\}$.

Proposition 4.2 *Let G be a binary tree labeled by $[\![1, n]\!]$ and denote by $\mathcal{F}$ the set of its non-terminal vertices. The following two conditions are equivalent*:

(1) *There exist vectors $w_1, \dots, w_{n-1} \in \mathbb{Z}^n$ such that G is compatible with $W = \{w_1, \dots, w_{n-1}\}$, and $\ker(\psi) = \mathbb{Z}W$.*

(2) *For all $\mathbf{v} \in \mathcal{F}$,*

$$\frac{\gcd(d_j,\, j \in \ell_1[\mathbf{v}])\gcd(d_j,\, j \in \ell_2[\mathbf{v}])}{\gcd(d_j,\, j \in \ell_1[\mathbf{v}] \cup \ell_2[\mathbf{v}])} \in \mathbb{N}\{d_j\,,\, j \in \ell_1[\mathbf{v}]\} \cap \mathbb{N}\{d_j\,,\, j \in \ell_2[\mathbf{v}]\}.$$

Proof. Let $\mathbf{v}$ be the root of G, and consider G_1 and G_2, the two components of the digraph obtained from G by removing the vertex $\mathbf{v}$ and the two edges leaving $\mathbf{v}$. We may assume that $\ell_1[\mathbf{v}] = [\![1, r]\!]$ and $\ell_2[\mathbf{v}] = [\![r+1, n]\!]$ for some $1 \le r \le n-1$. Then G_1 and G_2 are binary trees labeled by $[\![1, r]\!]$ and $[\![r+1, n]\!]$. The result is clear if $n = 2$. We will prove both implications by induction on n.

(1) ⇒ (2): Reindexing the w_i's if necessary, we may assume that w_{n-1} is the element of W associated to $\mathbf{v}$ through the map that makes G compatible with W,

and that $W_1 = \{w_1, \ldots, w_{r-1}\}$ and $W_2 = \{w_r, \ldots, w_{n-2}\}$ are the set of vectors in W such that G_i is compatible with W_i for $i = 1, 2$. There is a decomposition $\mathbb{Z}^n = \mathbb{Z}^r \oplus \mathbb{Z}^{n-r}$, where $\mathbb{Z}^r := \mathbb{Z}^r \times \{0\}^{n-r}$ and $\mathbb{Z}^{n-r} := \{0\}^r \times \mathbb{Z}^{n-r}$. Consider the linear map $\overline{\psi}_1 : \mathbb{Z}^n \to \mathbb{Z}$ induced by $\overline{\psi}_1(e_i) = d_i$ if $1 \leq i \leq r$ and $\overline{\psi}_1(e_i) = 0$ if $r < i \leq n$, and the map $\overline{\psi}_2 = \psi - \overline{\psi}_1$. Let ψ_1 (resp. ψ_2) be the restriction of $\overline{\psi}_1$ (resp. $\overline{\psi}_2$) to $\mathbb{Z}^r$ (resp. $\mathbb{Z}^{n-r}$). We claim that $\ker(\psi_1) = \mathbb{Z}W_1$ and $\ker(\psi_2) = \mathbb{Z}W_2$. By symmetry it suffices to prove the first equality. Clearly one has $\mathbb{Z}W_1 \subset \ker(\psi_1)$ because $\mathrm{supp}(w_i) \subset [\![1, r]\!]$ for $1 \leq i \leq r$. To show the reverse inclusion take $\alpha \in \ker(\psi_1) \subset \mathbb{Z}^r$. Since $\alpha \in \ker(\psi) = \mathbb{Z}W$ we can write

$$\alpha = (\lambda_1 w_1 + \cdots + \lambda_{r-1} w_{r-1}) + (\lambda_r w_r + \cdots + \lambda_{n-2} w_{n-2}) + \lambda_{n-1} w_{n-1} \ (\lambda_i \in \mathbb{Z}).$$

Hence $0 = \psi_1(\alpha) = \overline{\psi}_1(\alpha) = \lambda_{n-1}\overline{\psi}_1(w_{n-1}) = \lambda_{n-1}\overline{\psi}_1(w_{n-1}^+)$. In the last equality we use $\mathrm{supp}(w_{n-1}^+) \subset [\![1, r]\!]$ and $\mathrm{supp}(w_{n-1}^-) \subset [\![r+1, n]\!]$. As $\overline{\psi}_1(w_{n-1}^+) \neq 0$ we get $\lambda_{n-1} = 0$. Therefore $\lambda_r w_r + \cdots + \lambda_{n-2} w_{n-2} = 0$. This prove that $\alpha \in \mathbb{Z}W_1$, as required. Set

$$\begin{aligned} d &= \gcd(d_1, \ldots, d_n), \quad & d' &= \gcd(d_1, \ldots, d_r), \quad & d'' &= \gcd(d_{r+1}, \ldots, d_n), \\ \underline{d} &= \{d_1, \ldots, d_n\}, & \underline{d}' &= \{d_1, \ldots, d_r\}, & \underline{d}'' &= \{d_{r+1}, \ldots, d_n\}. \end{aligned}$$

Using induction and the claim we need only show $(d'd'')/d \in \mathbb{N}\underline{d}' \cap \mathbb{N}\underline{d}''$. For $1 \leq j \leq r$ and $r+1 \leq k \leq n$ we can write

$$\frac{d_k}{d} e_j - \frac{d_j}{d} e_k = \lambda_{jk}^1 w_1 + \cdots + \lambda_{jk}^{r-1} w_{r-1} + \lambda_{jk}^r w_r + \cdots + \lambda_{jk}^{n-2} w_{n-2} + \lambda_{jk}^{n-1} w_{n-1},$$

for some $\lambda_{kj}^1, \ldots, \lambda_{kj}^{n-1}$ in $\mathbb{Z}$. We write the last vector in W as $w_{n-1} = w_{n-1}^+ - w_{n-1}^-$ and $w_{n-1} = (a_1, \ldots, a_r, -a_{r+1}, \ldots, -a_n)$. Hence we get

$$\begin{aligned} -(d_j e_k)/d &= \lambda_{jk}^r w_r + \cdots + \lambda_{jk}^{n-2} w_{n-2} - \lambda_{jk}^{n-1} w_{n-1}^- \quad \Rightarrow \\ (d_j d_k)/d &= \lambda_{jk}^{n-1}(a_{r+1} d_{r+1} + \cdots + a_n d_n). \end{aligned}$$

Set $h = a_{r+1} d_{r+1} + \cdots + a_n d_n$. If we fix k and vary j, we get

$$\gcd\left((d_1 d_k/d), \ldots, (d_r d_k)/d\right) = \mu_k h \quad (\mu_k \in \mathbb{Z}) \ \Rightarrow \ d_k d' = \mu_k h d.$$

Therefore varying k yields $\gcd(d_{r+1}d', \ldots, d_n d') = hd$, $\mu \in \mathbb{Z}$. As a consequence $(d'd'')/d = (hd\mu)/d \in \mathbb{N}\underline{d}''$. A symmetric argument gives $(d'd'')/d \in \mathbb{N}\underline{d}'$, as required.

(2) $\Rightarrow$ (1): By induction there are $W_1 = \{w_1, \ldots, w_{r-1}\}$, $W_2 = \{w_r, \ldots, w_{n-2}\}$ such that G_i is compatible with W_i and $\ker(\psi_i) = \mathbb{Z}W_i$. The result will be proved if we give $w_{n-1} \in \mathbb{Z}^n$ such that $\mathrm{supp}(w_{n-1}^+) \subset [\![1, r]\!]$, $\mathrm{supp}(w_{n-1}^-) \subset [\![r+1, n]\!]$, and $\ker(\psi) = \mathbb{Z}W$ for $W = W_1 \cup W_2 \cup \{w_{n-1}\}$. By hypothesis,

$$(d'd'')/d = a_1 d_1 + \cdots + a_r d_r = a_{r+1} d_{r+1} + \cdots + a_n d_n,$$

where $a_i \in \mathbb{N}$ for all i. Setting $w_{n-1} := (a_1, \ldots, a_r, -a_{r+1}, \ldots, -a_n)$, one has that $\mathrm{supp}(w_{n-1}^+) \subset [\![1, r]\!]$ and $\mathrm{supp}(w_{n-1}^-) \subset [\![r+1, n]\!]$, and hence G is compatible with $W := W_1 \cup W_2 \cup \{w_{n-1}\}$. To complete the proof it remains to prove the equality $\mathbb{Z}W = \ker(\psi)$. Clearly $\mathbb{Z}W \subset \ker(\psi)$. To prove the reverse containment define $\sigma_{jk} = (d_j/d)e_k - (d_k/d)e_j$, $j, k \in [\![1, n]\!]$. By [13, Corollary 10.1.10] the set $\{\sigma_{jk} | \, j, k \in [\![1, n]\!]\}$ generates $\ker(\psi)$. Thus we need only show that $\sigma_{jk} \in \mathbb{Z}W$ for all $j, k \in [\![1, n]\!]$. If $j, k \in [\![1, r]\!]$ or $j, k \in [\![r+1, n]\!]$, then $\sigma_{jk} \in \ker(\psi_1) \subset \mathbb{Z}W$ or $\sigma_{jk} \in \ker(\psi_2) \subset \mathbb{Z}W$. Assume $j \in [\![1, r]\!]$ and $k \in [\![r+1, n]\!]$. From the equalities

$$
\begin{aligned}
S_1 &= \sum_{i=1}^{r} a_i \big((d_i/d')e_j - (d_j/d')e_i\big) = (d''/d)e_j - (d_j/d') \sum_{i=1}^{r} a_i e_i, \\
S_2 &= \sum_{i=r+1}^{n} a_i \big((d_i/d'')e_k - (d_k/d'')e_i\big) = (d'/d)e_k - (d_k/d'') \sum_{i=r+1}^{n} a_i e_i
\end{aligned}
$$

we conclude

$$(d_k/d'')S_1 - (d_j/d')S_2 = ((d_k/d)e_j - (d_j/d)e_k) - (d_j d_k/d'd'')w_{n-1}.$$

Since $S_i \in \ker(\psi_i) \subset \mathbb{Z}W$ we obtain $\sigma_{jk} \in \mathbb{Z}W$, as required. $\square$

Theorem 4.3 *The toric ideal P is a complete intersection if and only if there is a binary tree G labeled by $[\![1, n]\!]$ such that, for all non-terminal vertex $\mathbf{v}$ of G, one has that*

$$\frac{\gcd(d_j,\, j \in \ell_1[\mathbf{v}]) \gcd(d_j,\, j \in \ell_2[\mathbf{v}])}{\gcd(d_j,\, j \in \ell_1[\mathbf{v}] \cup \ell_2[\mathbf{v}])} \in \mathbb{N}\{d_j,\, j \in \ell_1[\mathbf{v}]\} \cap \mathbb{N}\{d_j,\, j \in \ell_2[\mathbf{v}]\}.$$

Proof. $\Rightarrow$) There are binomials $g_1, \ldots, g_{n-1}$ such that $P = (g_1, \ldots, g_{n-1})$. We may assume that $g_i = x^{\alpha_i} - x^{\beta_i}$ and $\mathrm{supp}(x^{\alpha_i}) \cap \mathrm{supp}(x^{\beta_i}) = \emptyset$ for all i. By Proposition 2.5(b) and Theorem 3.7 there exists a binary tree G labeled by $[\![1, n]\!]$ which is compatible with $\{g_1, \ldots, g_{n-1}\}$. Then G is compatible with $W = \{\widehat{g}_1, \ldots, \widehat{g}_{n-1}\}$ and $\ker(\psi) = \mathbb{Z}\{\widehat{g}_1, \ldots, \widehat{g}_{n-1}\}$ (see Proposition 2.5(a)). Thus applying Proposition 4.2 we obtain the required conditions.

$\Leftarrow$) By Proposition 4.2 there is $W = \{w_1, \ldots, w_{n-1}\} \subset \mathbb{Z}^n$ such that W is compatible with G and $\ker(\psi) = \mathbb{Z}W$. Setting $g_i := x^{w_i^+} - x^{w_i^-}$, one has that G is compatible with $\{g_1, \ldots, g_{n-1}\}$, and hence, using Theorem 3.7, we get

$$V(g_1, \ldots, g_{n-1}, x_i) = \{0\} \qquad (i = 1, \ldots, n).$$

Therefore by Proposition 2.5 we deduce the equality $P = (g_1, \ldots, g_{n-1})$. $\square$

Using a different approach, Delorme characterizes in [3] toric ideals of affine monomial curves that are complete intersections using a tool that he calls *suites distinguées* ([3, Lemme 8]). He then deduces his main result that can also be obtained from our characterization in terms of binary trees:

Corollary 4.4 ([3, Proposition 9]) *Assume that* $\gcd(\underline{d}) = 1$. *Then* P *is a complete intersection if and only if, reindexing the* d_i*'s if necessary, there exists* $r \in \{1, \ldots, n-1\}$ *such that, setting* $d' := \gcd(d_1, \ldots, d_r)$, $d'' := \gcd(d_{r+1}, \ldots, d_n)$, *and* $d'_i := \begin{cases} \frac{d_i}{d'} & \text{if } 1 \leq i \leq r \\ \frac{d_i}{d''} & \text{if } r+1 \leq i \leq n \end{cases}$, *one has that:*

(a) $d' \in \mathbb{N}\{d'_{r+1}, \ldots, d'_n\}$, $d'' \in \mathbb{N}\{d'_1, \ldots, d'_r\}$, *and*

(b) *the two toric ideals* $P_1 \subset k[x_1, \ldots, x_r]$ *and* $P_2 \subset k[x_{r+1}, \ldots, x_n]$ *defined by* $\{d_1, \ldots, d_r\}$ *and* $\{d_{r+1}, \ldots, d_n\}$ *respectively, are both complete intersections.*

Proof. $\Rightarrow$) If P is a complete intersection and $\mathbf{v}$ is the root of the binary tree G given by Theorem 4.3, we may assume, reindexing the d_i's if necessary, that $\ell_1[\mathbf{v}] = [\![1, r]\!]$ and $\ell_2[\mathbf{v}] = [\![r+1, n]\!]$ for some $r \in \{1, \ldots, n-1\}$. Setting $d' := \gcd(d_1, \ldots, d_r)$ and $d'' := \gcd(d_{r+1}, \ldots, d_n)$, one gets that $d'd'' \in \mathbb{N}\{d_1, \ldots, d_r\} \cap \mathbb{N}\{d_{r+1}, \ldots, d_n\}$ by Theorem 4.3, and (a) follows. Moreover, using the two binary subtrees of G obtained by removing $\mathbf{v}$ and the two edges leaving $\mathbf{v}$, one gets that (b) holds by applying Theorem 4.3.

$\Leftarrow$) Conversely, if P_1 and P_2 are complete intersections, let G_1 and G_2 be the two binary trees given by Theorem 4.3, denote by $\mathbf{v}_1$ and $\mathbf{v}_2$ their roots, and consider the binary tree G obtained by adding a vertex $\mathbf{v}$ and two edges leaving $\mathbf{v}$, one entering $\mathbf{v}_1$, the other entering $\mathbf{v}_2$. By (a), the vertex $\mathbf{v}$ of G (which is its root) satisfies the relation in Theorem 4.3, and any other non-terminal vertex of G satisfies it for being a non-terminal vertex of either G_1 or G_2, and hence P is a complete intersection. □

Remark 4.5 Given $d_1, \ldots, d_n$ such that P is a complete intersection, a binary tree G labeled by $[\![1, n]\!]$ such that the arithmetical conditions of Theorem 4.3 are satisfied encodes the following information:

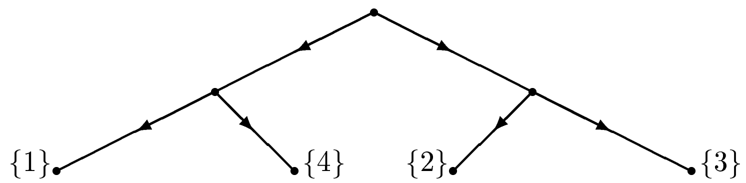
(i) The generators $\{g_1, \ldots, g_{n-1}\}$ of P and their degrees $D_1, \ldots, D_{n-1}$ can be obtained as shown in the proofs of Proposition 4.2 and Theorem 4.3.

(ii) The Frobenius number $g(S)$ of the numerical semigroup $S = \mathbb{N}\underline{d}$, that is the largest integer not in S, can be expressed entirely in terms of $\{d_1, \ldots, d_n\}$ when $\gcd(d_1, \ldots, d_n) = 1$.

This last assertion is a consequence of the following. Recall that the quasi-homogeneous Hilbert series of R/P is $H_P(z) = \frac{f(z)}{(1-z^{d_1})\cdots(1-z^{d_n})}$ for some polynomial $f \in \mathbb{Z}[z]$. When $\gcd(d_1, \ldots, d_n) = 1$, using that $R/P \simeq k[\Gamma]$, one can easily check that $H_P(z) = \frac{h(z)}{1-z}$ for some polynomial $h \in \mathbb{Z}[z]$ of degree $g(S)+1$. If P is a complete intersection, it is well-known that $f(z) = (1-z^{D_1}) \cdots (1-z^{D_{n-1}})$ where $D_1, \ldots, D_{n-1}$ are the degrees of the minimal quasi-homogeneous generators of P, and hence $g(S) = D_1 + \cdots + D_{n-1} - (d_1 + \cdots + d_n)$. Denoting by $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ the set of non terminal vertices of G and using (i), one gets the following formula:

$$g(S) = \left(\sum_{i=1}^{n-1} \frac{\gcd(d_j,\, j \in \ell_1[\mathbf{v}_i]) \gcd(d_j,\, j \in \ell_2[\mathbf{v}_i])}{\gcd(d_j,\, j \in \ell_1[\mathbf{v}_i] \cup \ell_2[\mathbf{v}_i])} \right) - \left(\sum_{i=1}^{n} d_i \right).$$

Example 4.6 Let k be an arbitrary field, and consider $d_1 = 16$, $d_2 = 27$, $d_3 = 45$ and $d_4 = 56$. The corresponding toric ideal $P \subset k[x_1, x_2, x_3, x_4]$ is a complete intersection because using the following binary tree labeled by $[\![1, 4]\!]$,



the arithmetical conditions in Theorem 4.3 are satisfied:

$$\begin{array}{rclcl} 112 = \dfrac{(16)(56)}{\gcd(16,56)} & \in & 16\mathbb{N} \cap 56\mathbb{N} & : & 2(56) \overset{(1)}{=} 7(16) \\ 135 = \dfrac{(27)(45)}{\gcd(27,45)} & \in & 27\mathbb{N} \cap 45\mathbb{N} & : & 3(45) \overset{(2)}{=} 5(27) \\ 72 = \dfrac{\gcd(16,56)\gcd(27,45)}{\gcd(16,27,45,56)} & \in & \{16,56\}\mathbb{N} \cap \{27,45\}\mathbb{N} & : & \\ & & & & 1(16) + 1(56) \overset{(3)}{=} 1(27) + 1(45). \end{array}$$

Moreover, the equalities (1), (2) and (3) provide, by Remark 4.5(i), a set of minimal generators of P:

$$g_1 = x_4^2 - x_1^7, \quad g_2 = x_3^3 - x_2^5, \quad g_3 = x_1x_4 - x_2x_3\,.$$

Finally, by Remark 4.5(ii), the Frobenius number of the numerical semigroup $S = \mathbb{N}\{16, 27, 45, 56\}$ is

$$g(S) = 112 + 135 + 72 - (16 + 27 + 45 + 56) = 175\,.$$

Remark 4.7 Toric ideals of affine monomial curves that are complete intersections were originally studied by Herzog in his paper [9]. In [9, Proposition 2.1], he considers the special situation where, after reindexing the d_i's if necessary, one has that

$$\frac{\gcd(d_1, \ldots, d_i)\, d_{i+1}}{\gcd(d_1, \ldots, d_{i+1})} \in \mathbb{N}\{d_1, \ldots, d_i\}\,,\ \forall i \in \{1, \ldots, n-1\}\,, \tag{1}$$

and he wonders in the next remark if this property charaterizes the complete intersection case. The answer to this question is negative, this was first observed by K. Watanabe in [14, Remark 1, p. 105]. In terms of binary trees, the situation in (1) corresponds to the case where $\#(\ell_2[\mathbf{v}]) = 1$ for each non-terminal vertex $\mathbf{v}$ of the binary tree involved in Theorem 4.3. Noting that in Theorem 4.3, one only needs to consider binary trees satisfying that $\#(\ell_1[\mathbf{v}]) \geq \#(\ell_2[\mathbf{v}])$ for any non-terminal vertex $\mathbf{v}$, it easily follows that, when $n = 3$, P is a complete intersection if and only (1) holds after a suitable reindexing of the d_i's. This does not occur when $n \geq 4$. When $n = 4$, one has two possible binary trees satisfying that $\#(\ell_1[\mathbf{v}]) \geq \#(\ell_2[\mathbf{v}])$ for any non-terminal vertex $\mathbf{v}$, and one can check that in Example 4.6, there is no way of indexing the d_i's so that (1) hold. Indeed, for $n \geq 1$, the number τ_n of binary trees with n terminal vertices and satisfying that $\#(\ell_1[\mathbf{v}]) \geq \#(\ell_2[\mathbf{v}])$ for any non-terminal vertex $\mathbf{v}$, is given by the following inductive formula:

$$\tau_1 = \tau_2 = 1 \text{ and, for all } n \geq 3\,,\ \tau_n = \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \tau_j \tau_{n-j}\,.$$

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