Normality Criteria for Monomial Ideals

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Conference Analysis Geometry and Beyond
Dedicated to the Memory of Prof. Gaetana Restuccia

Accademia Peloritana dei Pericolanti, Messina
October 20-22, 2025
Tuesday, October 21, 14:30 - 15:10

Let $R = K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. We use the following multi-index notation to denote the *monomials* of R:

$$x^a := x_1^{a_1} \cdots x_n^{a_n} \text{ for } a = (a_1, \ldots, a_n) \in \mathbb{N}^n.$$

Definition

An ideal I of R is called a *monomial ideal* if I is generated by a finite set of monomial $x^{v_1}, \ldots, x^{v_q}$, that is,

$$I = (x^{v_1}, \ldots, x^{v_q}) \subset R.$$

The set of monomials of R, denoted by $\mathbb{M}_n$, is a poset $(\mathbb{M}_n, \preceq)$ under divisibility, that is $x^a \preceq x^b$ if $x^b = x^\gamma x^a$ for some x^γ.

A monomial ideal I is always minimally generated by a unique finite set of monomials $\{x^{v_1}, \ldots, x^{v_q}\}$. This unique set of generators of I is denoted by $G(I)$.

An *anti-chain* of $(\mathbb{M}_n, \preceq)$ is a set of non-comparable monomials which is necessarily finite by Dickson's lemma.

There is a one-to-one correspondence between the set of monomial ideals and the set of anti-chains of $(\mathbb{M}_n, \preceq)$:

$$I \longmapsto G(I),$$

and for this reason a monomial ideal is also called an *anti-chain* ideal.

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let I be an ideal of R, an element $f \in R$ is *integral* over I if f satisfies an equation of the form

$$f^\ell + a_1 f^{\ell-1} + \cdots + a_{\ell-1} f + a_\ell = 0, \ a_k \in I^k.$$

The *integral closure* of I is the set of all elements $f \in R$ which are integral over I. The integral closure of I is an ideal of R and is denoted by $\overline{I}$.

If $I = \overline{I}$, I is said to be *complete* or *integrally closed*.

If $I^k = \overline{I^k}$ for all $k \geq 1$, the ideal I is said to be *normal*.

The *Newton polyhedron* of the ideal $I = (x^{v_1}, \ldots, x^{v_q}) \subset R$, denoted by $\mathrm{NP}(I)$, is the integral polyhedron given by

$$\mathrm{NP}(I) := \mathbb{R}^n_+ + \mathrm{conv}(v_1, \ldots, v_q).$$

Proposition

Let $I \subset R$ be a monomial ideal, and let $\mathrm{NP}(I)$ be its Newton polyhedron. Then

(a) $\overline{I^k} = (\{x^\alpha \mid (x^\alpha)^m \in I^{mk} \text{ for some } m \geq 1\})$ for all $k \geq 1$,

(b) $\overline{I^k} = (\{x^\alpha | \alpha/k \in \mathrm{NP}(I)\})$ for all $k \geq 1$.

Let $I = (x^{v_1}, \ldots, x^{v_q}) \subset R$ be a monomial ideal and let

$$\mathcal{A}' := \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\} \subset \mathbb{N}^{n+1},$$

where e_i is the ith unit vector. Let t be a new variable.

The Rees algebra of I can be written as

$$\begin{aligned} R[It] &= K[\{x_1, \ldots, x_n, x^{v_1}t, \ldots, x^{v_q}t\}] = K[\mathbb{N}\mathcal{A}'] \\ &= K[\{x^a t^b \mid (a, b) \in \mathbb{N}\mathcal{A}'\}] \\ &= R \oplus It \oplus I^2t^2 \oplus \cdots \oplus I^i t^i \oplus \cdots . \end{aligned}$$

The integral closure of $R[It]$ in its field of fractions is:

$$\begin{aligned} \overline{R[It]} &= K[\mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'] = K[\{x^a t^b \mid (a, b) \in \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'\}] \\ &= R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \cdots \oplus \overline{I^i}t^i \oplus \cdots , \end{aligned}$$

where $\mathbb{R}_+\mathcal{A}'$ is the cone generated by $\mathcal{A}'$.

Comparing $R[It]$ and $\overline{R[It]}$ in the equations above one has:

Proposition

Any of the following conditions are equivalent.

(a) $I^i = \overline{I^i}$ for all $i \geq 1$, that is, I is a normal ideal.

(b) $R[It]$ is a normal domain.

(c) $K[\mathbb{N}\mathcal{A}'] = K[\mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}']$.

(d) $\mathbb{N}\mathcal{A}' = \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$, that is, $\mathcal{A}'$ is a Hilbert basis.

The software program *Normaliz* computes the Hilbert basis of a rational cone. Thus we can use *Normaliz* to determine whether or not a monomial ideal is normal.

Example

Let $R = K[x_1, x_2]$ and $I = (x_1^3,\ x_2^4,\ x_1x_2^3)$. Note that

$$R[It] = K[x_1, x_2, x_1^3t, x_2^4t, x_1x_2^3t]$$

To compute $\overline{R[It]}$ we consider an input file for Normaliz:

amb_space 3
rees_algebra 3
3 0
0 4
1 3

Running normaliz we get

$$\overline{R[It]} = R[It][x_1^2x_2^2t],$$

that is, I is not normal and $I \neq \bar{I}$.

Normal monomial ideals are closed under taking minors, that is, if I is a normal monomial ideal and we make any variable x_i equal to 1 or 0, the resulting ideal is normal.

Theorem (Normality criterion for monomial ideals)

Let $I = (x^{v_1}, \ldots, x^{v_q})$ be a monomial ideal of R and let J_i be the ideal of R generated by all monomials obtained from $\{x^{v_1}, \ldots, x^{v_q}\}$ by making $x_i = 1$. Then I is normal if and only if

(a) *J_i is normal for all i and*

(b) *$\overline{I^r} \cap (I^r : \mathfrak{m}) = I^r$ for all $r \geq 1$, where $\mathfrak{m} = (x_1, \ldots, x_n)$.*

Let $R = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be a ring of polynomials over a field K.

Given k, r, s, t in $\mathbb{N}$ such that $k + r = s + t$, the *ideal of mixed products* is the square-free monomial given by:

$$L := I_k J_r + I_s J_t,$$

where

I_k is the ideal of R generated by the square-free monomials of degree k in the variables $x_1, \dots, x_m$, and

J_r is the ideal of R generated by the square-free monomials of degree r in the variables $y_1, \dots, y_n$.

The following classifies the normal mixed product ideals:

Theorem (G. Restuccia, - , Comm. Algebra, 2001)

If L is an ideal of mixed products and $L \neq R$, then L is normal if and only if it can be written (up to permutation of k, s and r, t) in one of the following forms:

(a) $L = I_k J_r + I_{k+1} J_{r-1}$, $k \geq 0$ and $r \geq 1$.

(b) $L = I_k J_r$, $k \geq 1$ or $r \geq 1$.

(c) $L = I_k J_r + I_s J_t$, $0 = k < s = m$, or $0 = t < r = n$, or $k = t = 0$, $s = 1$.

Making $r = 0$ in (b), we get $L = I_k J_0 = I_k R = I_k$, that is, we recover the k-th square-free Veronese ideal.

Definition

Let G be a graph with vertices $x_1, \ldots, x_n$ and let $t \geq 2$ be an integer. The *path ideal* of G, denoted by $I_t(G)$, is the ideal of $K[x_1, \ldots x_n]$ generated by all square-free monomials $x_{i_1} \cdots x_{i_t}$ such that the x_{i_j} is adjacent to $x_{i_{j+1}}$ for all $1 \leq j \leq t-1$.

Corollary

If G is a complete bipartite graph, then the path ideal $I_t(G)$ is normal for all $t \geq 2$.

Professor Gaetana Restuccia wrote some other papers on the subject of mixed product ideals:

- M. La Barbiera and G. Restuccia, Mixed product ideals generated by *s*-sequences, Algebra Colloq. (2011).

- G. Restuccia, Z. Tang, and R. Utano, Stanley conjecture on monomial ideals of mixed products, J. Commut. Algebra (2015).

- M. La Barbiera, M. Lahyane, and G. Restuccia, The Jacobian dual of certain mixed product ideals, Algebra Colloq. (2020).

And she was interested in symmetric and Rees algebras and strongly Koszul algebras, among many other things.

In Arch. Math., 2014, Herzog, Moghimipor, and Yassemi, introduce generalized mixed product ideals, which extend the construction of mixed product ideals and work of Bayati and Herzog.

Some of the properties and invariants that have been studied for mixed product ideals and generalized mixed product ideals include:

- Betti numbers,
- Regularity,
- Normality of Rees algebras,
- Unmixed and Cohen–Macaulay properties,
- Invariants of symmetric algebras,
- Persistence property of associated primes.

Here is a list of papers in chronological order where these properties and invariants are studied:

- G. Rinaldo, Betti numbers of mixed product ideals, Arch. Math. (2008).

- C. Ionescu and G. Rinaldo, Some algebraic invariants related to mixed product ideals, Arch. Math. (2008).

- P. L. Stagliano, Powers of mixed products ideals and normality of the associated monomial algebra, Rend. Circ. Mat. Palermo (2008).

- L. T. Hoa, and N. D. Tam, On some invariants of a mixed product of ideals, Arch. Math. (2010).

- J. Herzog, R. Moghimipor, and S. Yassemi, Generalized mixed product ideals, Arch. Math. (2014).

- G. Rinaldo, Sequentially Cohen-Macaulay mixed product ideals, Algebra Colloq. (2015).

- L. Chu and J. Perez, The Stanley regularity of complete intersections and ideals of mixed products, J. Algebra Appl. (2017).

- R. Moghimipor and A. Tehranian, Linear resolutions of powers of generalized mixed product ideals, Iran. J. Math. Sci. Inform. (2019).

- R. Moghimipor, On the normality of generalized mixed product ideals, Arch. Math. 115 (2020).

- R. Moghimipor, Algebraic and homological properties of generalized mixed product ideals, Arch. Math. (2020).

- R. Moghimipor, On the Cohen-Macaulayness of bracket powers of generalized mixed product ideals, Acta Math. Vietnam. (2022).

- R. Moghimipor, Strong persistence property of generalized mixed product ideals, Bull. Math. Soc. Sci. Math. Roumanie (2023).

- M. La Barbiera and R. Moghimipor, Normality of Rees algebras of generalized mixed product ideals, Int. Electron. J. Algebra (2024).

Let $\mathcal{A} = \{v_1, \ldots, v_q\} \subset \mathbb{N}^n$ be the set of bases of a discrete polymatroid of rank d, that is,

(a) $|v_i| = d$ for all i, where $|v_i|$ =sum of the entries of v_i, and

(b) given any two $a = (a_i), c = (c_i)$ in $\mathcal{A}$, if $a_i > c_i$ for some index i, then there is an index j with $a_j < c_j$ such that $a - e_i + e_j$ is in $\mathcal{A}$.

The ideal $I = (x^{v_1}, \ldots, x^{v_q})$ is called a *polymatroidal ideal* of degree d.

Theorem

If I is a polymatroidal ideal, then I is normal.

Let $R = K[x_1, \ldots, x_n]$ be polynomial ring over a field K.

Given a sequence of integers $(s_1, s_2, \ldots, s_n; d)$ such that $1 \le s_j \le d \le \sum_{i=1}^n s_i$ for all j, let $\mathcal{A}$ be the set of partitions

$$\mathcal{A} = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid a_1 + \cdots + a_n = d;\ 0 \le a_i \le s_i \,\forall\, i\},$$

and let I be the ideal

$$I = (\{x^a \mid a \in \mathcal{A}\}).$$

The ideal I is called an ideal of *Veronese type* of *degree d* with defining sequence $(s_1, \ldots, s_n; d)$.

If $s_i = 1$ for all i, I is the d-th *square-free Veronese ideal*,

if $s_i = d$ for all i, I is the d-th *Veronese ideal*.

one has the following implications

$$I \text{ is the } d\text{-th square-free Veronese ideal} \Rightarrow$$
$$I \text{ is an ideal of Veronese type of degree } d \Rightarrow$$
$$I \text{ a polymatroidal ideal of degree } d.$$

I particular one has:

Corollary

(a) *If I is the d-th square-free Veronese ideal, then I is normal.*

(b) *If I is an ideal of Veronese type of degree d, then I is normal.*

Notation If $\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{R}^q$, we let $|\lambda| = \sum_{i=1}^{q} \lambda_i$.

Proposition (Linear Algebra Normality Criterion)

Let $I = (x^{v_1}, \ldots, x^{v_q})$ be a monomial ideal and let A be the matrix with column vectors $v_1, \ldots, v_q$. The following conditions are equivalent.

(a) *I is a normal ideal.*

(b) *For each pair of vectors $\alpha \in \mathbb{N}^n$ and $\lambda \in \mathbb{Q}_+^q$ such that $A\lambda \leq \alpha$, there is $m \in \mathbb{N}^q$ satisfying $Am \leq \alpha$ and $|\lambda| = |m| + \epsilon$ with $0 \leq \epsilon < 1$.*

Let $I = (x^{v_1}, \ldots, x^{v_q})$ be a monomial ideal and let A be the matrix with column vectors $v_1, \ldots, v_q$.

The linear system $x \geq 0; xA \geq \mathbf{1}$ has the *integer rounding property* if

$$\max\{\langle y, \mathbf{1}\rangle \mid y \in \mathbb{N}^q;\ Ay \leq \alpha\} = \lfloor \max\{\langle y, \mathbf{1}\rangle \mid y \geq 0;\ Ay \leq \alpha\} \rfloor$$

for each $\alpha \in \mathbb{N}^n$ for which the right-hand side is finite.

From the linear algebra normality criterion one obtains:

Corollary

The monomial ideal I is normal if and only if the system $x \geq 0; xA \geq \mathbf{1}$ has the integer rounding property.

Proposition (Linear Programming Membership Test)

Let $I = (x^{v_1}, \ldots, x^{v_q})$ be a monomial ideal of R, let A be the matrix with column vectors $v_1, \ldots, v_q$, and let x^α be a monomial in R. Then the following conditions are equivalent:

(a) $x^\alpha \in \overline{I^k}$, $k \geq 1$.

(b) $\min\{\langle \alpha, x\rangle \mid x \geq 0; xA \geq \mathbf{1}\} \geq k$.

Let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. The *edge ideal* of G, denoted by $I(G)$, is the ideal of R given by

$$I(G) := (\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}).$$

Let $A \subset V(G)$ be a set of vertices of G. The *neighbor set* of A, denoted $N_G(A)$, is the set of all vertices of G that are adjacent with at least one vertex of A.

In this part we present the normality criterion for the edge ideal $I(G)$ and for the Rees algebra $R[I(G)t]$.

Example

In an email communication, Hochster showed to Vasconcelos the first example of a connected graph G whose edge ideal is not normal:

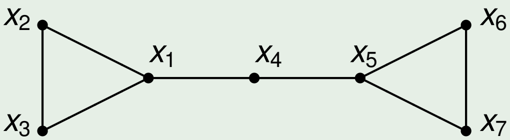

Figure: Hochster configuration.

Note that $f = (x_1x_2x_3)(x_5x_6x_7)$ satisfies $f \in \overline{I(G)^3} \setminus I(G)^3$.

This example leads to the following concept:

Definition

A *Hochster configuration* of G consists of two vertex disjoint induced odd cycles C_1, C_2 such that

$$C_1 \cap N_G(C_2) = \emptyset.$$

Theorem

The edge ideal $I(G)$ of a graph G is normal if and only if G admits no Hochster configurations.

Corollary

If G is a bipartite graph, then $I(G)$ is normal

Let G be a graph. A subset $C \subset V(G)$ is a *minimal vertex cover* of G if every edge of G contains at least one vertex of C and C is minimal with respect to inclusion.

The *ideal of covers* of I is given by

$$I_c(G) = (\{x_1^{a_1} \cdots x_n^{a_n} \mid \{x_i \mid a_i > 0\} \text{ is a minimal vertex cover}\}).$$

Theorem

If G is a perfect graph, then $I_c(G)$ is normal.

As a consequence if G is bipartite, then $I_c(G)$ is normal. If G is an odd cycle it is known that $I_c(G)$ is normal.

A main problem in this area is the characterization of the normality of the ideal of covers of a graph G in terms of the combinatorics of G and its complement $\overline{G}$.

This problem was solved for graphs with $\dim(R/I(G)) \leq 2$ in terms of Hochster configurations of $\overline{G}$.

Universita' di Messina, 1999.

THE END