Monomial Ideals and the Computation of Multiplicities

D. Delfino, A. Taylor[*], W. V. Vasconcelos[†], N. Weininger[‡]
Department of Mathematics
Rutgers University
110 Frelinghuysen Rd
Piscataway, New Jersey 08854-8019
e-mail: {delfino,ataylor,vasconce,nweining}@math.rutgers.edu

R. H. Villarreal[§]

Departamento de Matemáticas
Centro de Investigación y de Estudios Avanzados del IPN
Apartado Postal 14–740
07000 México City, D.F.
e-mail: vila@math.cinvestav.mx

Abstract

The theory of the integral closure of ideals has resisted direct approaches to some of its basic questions (membership and completeness tests, and construction). We mainly treat the membership problem in the monomial case by exploiting the connection with multiplicities and its linkage to the computation of volumes of polyhedra. We discuss several existent software packages and introduce our own contribution, a *Monte Carlo* based approach to the computation of volumes. Finally, we make comparisons of multiplicities of general ideals and of their initial ideals.

Introduction

Let R denote a Noetherian ring and I one of its ideals. The *integral closure* of I is the ideal $\overline{I}$ of all elements of R that satisfy an equation of the form

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0, \qquad a_i \in I^i.$$

[0]2000 AMS *Mathematics Subject Classification.* Primary 13B22; Secondary 13H15, 13P10.
[0]*Key Words and Phrases.* Integral closure of ideal, Monte Carlo method, multiplicity, volume of polyhedra.
[*]Partially supported by NSF-VIGRE grant.
[†]Partially supported by the NSF.
[‡]Partially supported by NSF-VIGRE grant.
[§]Partially supported by CONACyT and SNI, México.

There are several issues associated with this notion, from which we single out the following. Let $R = k[x_1, \ldots, x_n]$ be a ring of polynomials over the field k, let $I = (f_1, \ldots, f_m) \subset R$, and let $f \in R$. Our main concern is how to carry out the following tests/construction:

- *Membership Test*: $f \in \overline{I}$?
- *Completeness Test*: $I = \overline{I}$?
- *Construction Task*: $I \rightsquigarrow \overline{I}$?
- *Complexity Cost*: $\mathrm{cx}(I \rightsquigarrow \overline{I})$?

In the literature one does not find effective methods to generally deal with these problems. The difficulty arises, partly, from the specialized nature of the equations the elements need to satisfy. The exception, when we understand the problem fully, is the case of monomial ideals. In this case, $\overline{I}$ is the monomial ideal defined by the integral convex hull of the exponent vectors of I (see [8, p. 140]). Through the techniques of integer programming, all four problems can, theoretically and often in practice, be solved. For non-monomial ideals, only specialized cases of some of these questions have been dealt with ([7] treats the completeness test for generic complete intersections).

Our interest in these questions is reinforced by its connections to another issue, which has not been adequately dealt with either, the computation of multiplicities in local rings. If $(R, \mathfrak{m})$ is a Noetherian local ring of Krull dimension d, and I is an $\mathfrak{m}$–primary ideal then $e(I)$, the *multiplicity* of I, is the integer

$$\lim_{n\to\infty} \frac{\lambda(R/I^n)}{n^d} d!,$$

where $\lambda(\cdot)$ is the length function. The Hilbert function of the ideal is $\lambda(R/I^n)$, which is given by a polynomial of degree d for $n \gg 0$ (see [2], [8]). When the ideal is monomial, the limit can be interpreted as a Riemann sum of volumes (normalized by the factor $d!$) and we exploit this connection.

These numbers are not easily captured, if at all, by Gröbner bases computations. In part this is because a large number of indeterminates are required to frame the calculation. A simplified version occurs when I is the maximal ideal and a conversion to a monomial ideal is possible through a theorem of Macaulay (Theorem 4.1). The connection between the two sets of issues, integral closure and multiplicity, rests primarily on a well-known theorem of Rees ([15], and its generalizations): For $I \subset L$, $e(I) = e(L)$ if and only if $L \subset \overline{I}$.

We briefly describe our results. The first section is an elementary recasting of the description of $\overline{I}$ for a monomial ideal I. It is mainly used to recast the interpretation of multiplicity as a volume. It also exhibits the fact that the degrees of the generators of $\overline{I}$ do not exceed the top degree of a generating set of I by more than $d-1$. In some sense this solves the complexity count of the determination of $\overline{I}$ by placing a bounding box around I, according to Corollary 1.3

Equipped with the understanding of multiplicities of monomial ideals of finite co-length, in section 2 we introduce a *Monte Carlo* method for the computation of volumes of polytopes and report on our experience with it. It is simple to set up and we found it comparable (in deriving estimates) to the more technical approaches aimed at exact computation. One of our goals is to explore the existing library of software to deal with these questions. We are particularly interested in problems in large numbers of indeterminates, obviously beyond the horizon of symbolic computation engines based on Gröbner basis techniques.

In section 3, we use standard linear programming techniques to deal with the four tests above. Ideally, one would like to answer the first two tests through an oracle matrix. For instance, in the membership test: Given a monomial ideal I, there is a matrix $\mathbb{A}$ and a vector $\mathbf{b}$ such that a monomial $\mathbf{x}^{\mathbf{v}} \in \overline{I}$ if and only if

$$\mathbb{A} \cdot \mathbf{v} \geq \mathbf{b}.$$

We show how to do this with off-the-shelf software, and rather efficiently for ideals of finite co-length. For this class of ideals, we also show how any membership oracle can be used as a completeness test and as a path to the construction task using exclusively monomial arithmetic.

The last section is an exploration of the relationships between the multiplicities of an ideal I of finite co-length and of its initial ideal $in_>(I)$, for some term ordering. It always holds that $e(I) \leq e(in_>(I))$, with equality meaning that for each integer n, $in_>(I^n)$ is integral over $(in_>(I))^n$ (Theorem 4.3). Note that in this case, the initial algebra of the Rees algebra $R[It]$ is Noetherian (a very infrequent occurrence).

Regrettably, the methods developed to compute multiplicities and treat integral closure issues do not extend to general ideals of rings of polynomials, or to affine algebras. In these cases, one can still appeal to Gröbner bases methods for small-scale examples.

1 Integral closure of monomial ideals

To set up the framework, we recall some general facts about the integral closure of monomial ideals that are required for our treatment of multiplicity. Let $R = k[x_1, \ldots, x_n]$ be a ring of polynomials over the field k, and let I be the ideal generated by the set of monomials $\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_m}$. First we recall two descriptions of the integral closure of I.

One standard way to describe the integral closure of a monomial ideal is ([8, Exercise 4.23]):

Proposition 1.1 *Suppose $R = k[x_1, \ldots, x_n]$, and I is generated by a set of monomials $\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_m}$. Let Γ be the set of exponents of monomials in I,*

$$\Gamma = \bigcup_{i=1}^{m} v_i + \mathbb{N}^n.$$

Regarding Γ as a subset of $\mathbb{R}^n_+$, let Λ be the convex hull of $\mathbb{R}^n_+ + \Gamma$, and let Γ^* be the set of integral points in Λ. Then $\overline{I}$ is the ideal generated by $\mathbf{x}^v$, $v \in \Gamma^*$.

We will use a second description (see [20, Section 6.6], [21, Section 7.3]) of the generators of the integral closure. If $\mathbf{x}^v \in \overline{I}$, it will satisfy an equation

$$(\mathbf{x}^v)^\ell \in I^\ell,$$

and therefore we have the following equation for the exponent vectors,

$$\ell \cdot v = u + \sum_{i=1}^{m} r_i \cdot v_i, \quad r_i \geq 0, \quad \sum_{i=1}^{m} r_i = \ell.$$

This means that $v = \dfrac{u}{\ell} + \alpha$, where α belongs to the convex hull $\mathrm{Conv}(v_1, \ldots, v_m)$ of $v_1, \ldots, v_m$. The vector v can be written as (set $w = \frac{u}{\ell}$)

$$v = \lfloor w \rfloor + (w - \lfloor w \rfloor) + \alpha,$$

and it is clear that the integral vector

$$v_0 = (w - \lfloor w \rfloor) + \alpha$$

also has the property that $\mathbf{x}^{v_0} \in \overline{I}$.

Proposition 1.2 *Let I be an ideal generated by the monomials $\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_m}$. Let C be the rational convex hull of $V = \{v_1, \ldots, v_m\}$ and*

$$B = [0, 1) \times \cdots \times [0, 1) = [0, 1)^n.$$

Then $\overline{I}$ is generated by $\mathbf{x}^v$, where $v \in (C + B) \bigcap \mathbb{N}^n$.

For simplicity we set $B(V) = (C + B) \bigcap \mathbb{N}^n$.

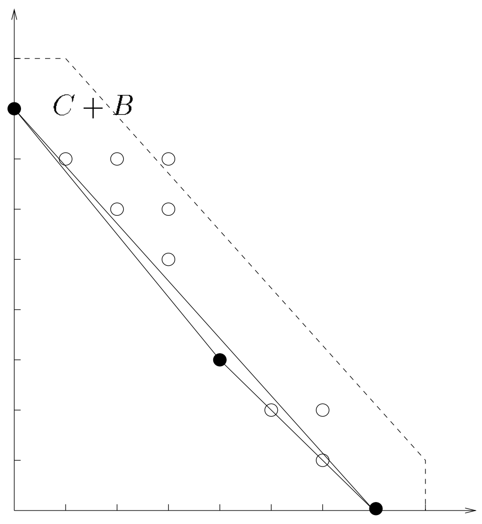

Figure 1: $B(V)$: The dotted lines indicate the boundary of $C + B$. The open circles are those lattice points which give elements in the integral closure of I. The lattice point that is not in $(C + B) \bigcap \mathbb{N}^n$, is in the ideal generated by the lattice points in $(C + B) \bigcap \mathbb{N}^n$.

The following help illustrate some of the issues with computing the integral closure of an ideal. First, a degree bound for the generators of the integral closure arises directly from Proposition 1.2. A sharper bound might depend on the codimension of the ideal.

Corollary 1.3 *Let I be a monomial ideal of $k[x_1, \ldots, x_n]$, generated by monomials of degree at most d. Then $\overline{I}$ is generated by monomials of degree at most $d+n-1$.*

The following example shows that the integral closure of a monomial ideal I, although by Proposition 1.1 defined by the integral convex hull of all the exponent vectors of I, may not be generated by the monomials defined by the integral convex hull of the exponent vectors of a minimal set of generators of I. The vector $\langle 3, 5\rangle$ in Figure 1 also illustrates this possibility. This, of course, makes the determination of $\overline{I}$ a great deal harder. We will revisit this example when we give two membership tests.

Example 1.4 Let I be the ideal of the ring of polynomials $R = k[x_1, \ldots, x_8]$ defined by the monomials given through exponent vectors $v_1, \ldots, v_8$:

$$\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}$$

Let $L = I^3$, and consider the vector $v = (2, \ldots, 2)$. In view of the equality

$$8v = (1, \ldots, 1) + 3(v_1 + \cdots + v_8),$$

one has

$$v = (\frac{1}{8}(3v_1) + \cdots + \frac{1}{8}(3v_8)) + (\frac{1}{8}, \ldots, \frac{1}{8}),$$

which shows that $\mathbf{x}^v$ lies in the integral closure of L; note that this monomial has degree 16 while L is generated by monomials of degree 15. Since the vectors v_i are linearly independent, one can easily check with *Maple* that decrementing v, in any coordinate, by 1 produces elements that do not lie in the convex hull of $\{3v_1, \ldots, 3v_8\}$. It follows that $\overline{L}$ requires minimal generators of degree at least 16.

We next recall the connection between integral closure, multiplicity and the computation of volumes of polyhedra. Let $f_1 = \mathbf{x}^{v_1}, \ldots, f_r = \mathbf{x}^{v_r}$ be a set of monomials generating the ideal I. The convex hull $C(V)$ of the v_i's partitions the positive quadrant into 3 regions: an unbounded connected region, $C(V)$ itself and the complement $\mathcal{P}$ of the other two. The bounded region $\mathcal{P}$ is the region most pertinent to our calculation (see also [18, p. 235], [19]).

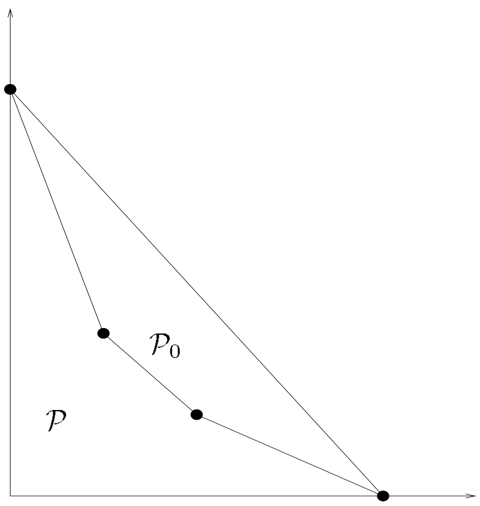

Figure 2: The polytope boundary is $C(V)$ and $\mathcal{P}$ is as marked. We will also refer to the polytope marked by $\mathcal{P}_0$ and the simplex that bounds $\mathcal{P}$ and $\mathcal{P}_0$ will be referred to as Δ.

The integral closure $\overline{I}$ is generated by the monomials whose exponents have the form

$$\sum_{i=1}^{r} r_i v_i + \epsilon \subset \mathbb{N}^d$$

such that $r_i \geq 0$ and $\sum r_i = 1$ and ϵ is a positive vector with entries in $[0,1)$. Suppose that I is of finite co-length, then, using the notation of Proposition 1.1, $\lambda(R/\overline{I})$ is the number set of lattice points not in $C(\mathbb{R}^n_+ + \Gamma)$.

Consider the integral closure of I^n. According to the valuative criterion ([22, p. 350]), $\overline{I^n}$ is equal to the integral closure of the ideal generated by the n^{th} powers of the f_i's. This means that the generators of $\overline{I^n}$ are defined by the exponent vectors of the form

$$\sum r_i n v_i + \epsilon,$$

with r_i and ϵ as above. We rewrite

$$n(\sum r_i v_i + \frac{\epsilon}{n})$$

so the vectors enclosed must have denominators dividing n. To deal with $\overline{I^n}$ we are going to use the set of vectors v_i, but change the scale by $1/n$. This means that each I^n determines the same $\mathcal{P}$. The length ℓ_n of $R/\overline{I^n}$ is the number of scaled lattice points in $\mathcal{P}$. Placing the lower left corner of a hypercube of side $1/n$ at each lattice point we see that the sum of the volumes of the hypercubes is equal to the number of lattice points times $(1/n)^d$ which in turn is $(1/n)^d \ell_n = (1/n)^d \lambda(R/\overline{I^n})$. However, this sum is also a Riemann sum approximating the volume of $\mathcal{P}$ and thus the limit of this quantity as $n \to \infty$ is just the exact volume of $\mathcal{P}$ (see Figure 3). This number, multiplied by $d!$, is the multiplicity of the ideal.

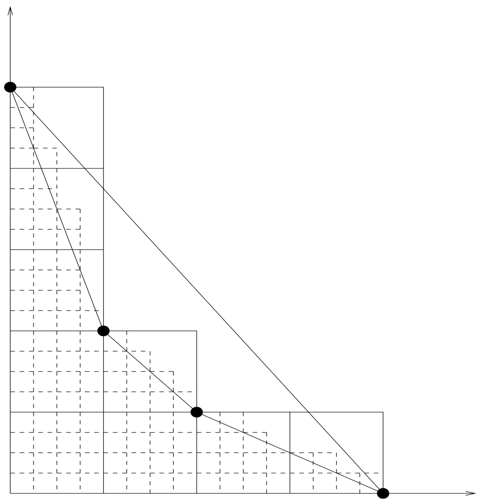

Figure 3: The cubes of side one and side $\frac{1}{4}$ are shown.

Let us sum up some of these relationships between multiplicities and volumes of polyhedra (see [19, p. 131]).

Proposition 1.5 *Let I be a monomial ideal of $R = k[x_1, \ldots, x_d]$ generated by $\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_m}$. Suppose that $\lambda(R/I) < \infty$. If $\mathcal{P}$ is the region of $\mathbb{N}^d$ defined by I then*

$$e(I) = d! \cdot \mathrm{Vol}(\mathcal{P}). \tag{1}$$

Example 1.6 Our first example is an ideal of $k[x, y, z]$. Suppose

$$I = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma), \quad \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} < 1.$$

The inequality ensures that the fourth monomial does not lie in the integral closure of the other three. A direct calculation shows that the multiplicity is indeed the volume of the region $\mathcal{P}$ times $d!$, which in this case is given by a nice formula

$$e(I) = ab\gamma + bc\alpha + ac\beta.$$

We observe that $\mathcal{P}$ is not a polytope, but can be expressed as the difference between two polytopes directly determined by the set of exponents vectors defining I, $V = \{v_1, \ldots, v_m\}$, $v_i \neq 0$. Since I has finite co-length, suppose the first d exponent vectors correspond to the generators of $I \cap k[x_i]$, $i = 1, \ldots, d$. Let Δ be the polyhedron defined by these vectors, $\Delta = C(0, v_1, \ldots, v_d)$, and denote by $\mathcal{P}_0$ the convex hull of V (see Figure 2). We note

$$\mathcal{P} = \Delta \setminus \mathcal{P}_0,$$

and therefore

$$\mathrm{Vol}(\mathcal{P}) = \mathrm{Vol}(\Delta) - \mathrm{Vol}(\mathcal{P}_0) = \frac{|v_1| \cdots |v_d|}{d!} - \mathrm{Vol}(\mathcal{P}_0).$$

We use this relationship to compute multiplicities. If we set

$$p = \frac{\mathrm{Vol}(\mathcal{P}_0)}{\mathrm{Vol}(\Delta)}$$

then the proposition follows.

Proposition 1.7 *Let I be a monomial ideal of finite colength generated by the monomials $\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_m}$. With the notation above, we have*

$$e(I) = (1 - p)|v_1| \cdots |v_d|.$$

2 A probabilistic approach to volumes and multiplicities

There is an extensive literature on the computation of volumes of polyhedra. We benefited from the discussion of volume computation in [4]. The associated costs of the various methods depend on how the convex sets are represented. They often require conversion from one representation to another. We propose a manner in which to approach the calculation of p, i.e. the calculation of $\mathrm{Vol}(\mathcal{P}_0)$ as a fraction of $\mathrm{Vol}(\Delta)$. First, note that Δ is defined by the equations

$$\Delta : \frac{x_1}{|v_1|} + \cdots + \frac{x_d}{|v_d|} \leq 1, \quad x_i \geq 0. \tag{2}$$

According to [6, pp. 284–285], since $\mathcal{P}_0$ is the convex hull of the vectors v_i, $i = 1, \ldots, m$, there are standard linear programming techniques to convert the convex hull description of $\mathcal{P}_0$ into an intersection of halfspaces

$$\mathcal{P}_0 : \mathbb{A} \cdot \mathbf{x} \leq \mathbf{b}. \tag{3}$$

Equally important, these linear programming techniques have been converted into very efficient routines in several programming environments. We will focus on those routines found in the collection [5].

Our statistical approach is based on classical *Monte Carlo* quadrature methods ([17]). Sampling a very large number of points in Δ, and checking when those points lie in $\mathcal{P}_0$ are both computationally straight forward because of the ease of the descriptions given in equations (2) and (3).

Our proposal consists of making a series of N independent trials, keeping track of the number of hits H, and using the frequency $\frac{H}{N}$ as an approximation for p. According to basic probability theory, these approximations come with an attached probability in the sense that for small $\epsilon > 0$

$$\text{Probability}\left\{\left|\frac{H}{N} - p\right| < \epsilon\right\}$$

is high. This estimation is based on Chebyshev's inequality ([10, p. 233]). We briefly review this inequality. If X is a random variable with finite second moment $\mathrm{E}(X^2)$, then for any $t > 0$

$$P\{|X| \geq t\} \leq t^{-2}E(X^2).$$

In particular for a variable X of mean $E(X) = \mu$ and finite variance $\mathrm{Var}(X)$, for any $t > 0$

$$P\{|X - \mu| \geq t\} \leq t^{-2}\mathrm{Var}(X). \tag{4}$$

For a set of N independent trials $x_1, \ldots, x_N$ of probability p, the random variable we are interested in is the average number of hits

$$X = \frac{x_1 + \cdots + x_N}{N} = \frac{H}{N}.$$

We have $E(X) = p$ and $\mathrm{Var}(X) = \sqrt{\frac{p(1-p)}{N}}$. If we set $\epsilon = t^{-2}\mathrm{Var}(X)$, and substitute into (4), we obtain

$$P\left\{\left|\frac{H}{N} - p\right| < \sqrt{\frac{p(1-p)}{\epsilon N}}\right\} > 1 - \epsilon.$$

Since $p(1-p) \leq \frac{1}{4}$, it becomes easy to estimate the required number of trials to achieve a high degree of confidence. Thus, for instance, a crude application shows that in order to obtain a degree of confidence of 0.95, and $\epsilon = 0.02$, the required number of trials should be $N \geq 12,500$ (Actually, a refined analysis, using the law of large numbers, cuts this estimate by $\frac{4}{5}$).

We have implemented this probabilistic approach to multiplicity. Our implementation uses off-the-shelf software. We illustrate our implementation though the discussion of some examples.

Example 2.1 Computing multiplicity using probability requires a conversion that uses PORTA (see [5]), a collection of transformation techniques in linear programming.

We will illustrate an application of the probabilistic method for the calculation of multiplicity in the setting of Proposition 1.5. Let $I = (x^3, y^4, z^5, w^6, xyzw)$. Proposition 1.5 gives $e(I) = 342$. To apply the probabilistic method, the exponents are written into a matrix and PORTA is used to obtain the inequalities defining the convex hull. The PORTA input and output are recorded below.

```
The points defining the convex hull must be written in a file with
the extension .poi [say mult1.poi] and the routine ``traf'' is called

traf mult1.poi

----------------------------------
The content of mult1.poi is:
----------------------------------
DIM = 4

CONV_SECTION
3 0 0 0
0 4 0 0
0 0 5 0
0 0 0 6
1 1 1 1

END
------------------------------------
The output file is the desired set of linear inequalities
and it is put in the file mult1.poi.ieq:
------------------------------------
DIM = 4

VALID
1 1 1 1

INEQUALITIES_SECTION
(  1) -23x1-15x2-12x3-10x4 <= -60
(  2) -20x1-15x2-12x3-13x4 <= -60
(  3) -10x1- 9x2- 6x3- 5x4 <= -30
(  4) - 4x1- 3x2- 3x3- 2x4 <= -12
(  5) +20x1+15x2+12x3+10x4 <=  60

END
```

A C^{++} program is then used to calculate the probability. Testing with 10,000 points gives a probability of .04989 and a multiplicity of 342.04.

Now we present more examples utilizing our implementation of our proposed probability based algorithm for computing the multiplicity of monomial ideals. We include an analysis of their run times and probable accuracy of results. In presenting these examples, the dimension-independence of the method is clear. However, the differences between theoretical results and implemented results are also clear. All results listed were obtained on a Pentium

III processor that runs at 900 MHZ, has 256 MB RAM and is operating under Red Hat Linux.

We illustrate the results of our algorithm using three examples. For the purposes of the examples, we will refer to our algorithm as *POLYPROB*. We revisit Example 2.1 and give two other examples for comparing MACAULAY2 [12], VINCI [4], NORMALIZ [3], and POLYPROB. VINCI is an alternate program for computing the volume of a polytope, while our computations in MACAULAY2 are classical, meaning we compute the leading coefficient of the Hilbert Polynomial of the associated graded ring.

Example 2.2 Our second example is again in a four dimensional ring

$$I = (x^4, y^5, z^6, w^7, xz^2w, y^2zw^2, xyzw).$$

This example is more complicated, but we can still use MACAULAY2, VINCI and POLYPROB to compute the multiplicity.

Example 2.3 Last we present an example where MACAULAY2 fails, and the issues of accuracy and speed in POLYPROB and VINCI are also illustrated. This example is sixteen dimensional

$$(x_1^2, x_2^3, x_3^4, x_4^5, x_5^6, x_6^7, x_7^8, x_8^9, x_9^{10}, x_{10}^{11}, x_{11}^{12}, x_{12}^{13}, x_{13}^{14}, x_{14}^{15}, x_{15}^{16}, x_{16}^{17}, x_3x_5x_8x_{10}x_{12}x_{14}x_{16},$$
$$x_2x_5x_7x_{13}, x_3x_7x_9x_{10}x_{13}^2x_{15}^2x_{16}, x_2x_4x_6x_{11}^2x_{14}^2, x_4x_6x_8x_{11}, x_1x_9, x_1x_{15}).$$

For this ideal, while we have

$$P\left\{\left|\frac{H}{N} - p\right| < \sqrt{\frac{p(1-p)}{\epsilon N}}\right\} > 1 - \epsilon,$$

when we multiply the probability by 17! to get the multiplicity, we also multiply the error by this same number. In Example 2.3 for $\epsilon = .02$ and $N = 20000$ we get $H = 16618$ and $H/N = .8309$ in one trial. The formula states that the probability that $|.8309 - p| < \frac{1}{4(.01)(20000)} = .000625$ is greater than .98. However, we can only say that $|(1 - .8309)17! - e(I)| < (.00125)(17!) = 2.22305(10^{11})$. Even with everything else the same and $N = 1,000,000$, $|.8309 - p| < .000025$, but $|(1 - .8309)17! - e(I)| < (.0000125)(17!) = 4.44609(10^9)$. We would need $N = 10^{14}$ to get the error on the multiplicity, using POLYPROB, to around 10. Unfortunately, the numbers we are dealing with mean that using standard floating point arithmetic there will be large computer precision error involved. The program VINCI also has this problem for large computations. We have been able to implement POLYPROB using GMP [11] aribitrary precision arithmetic and these are the numbers we include here. Unfortunately, it would take days to run 2.3 in POLYPROB with $N = 10^{14}$.

For each example, NORMALIZ computed the multiplicity (342, 546, and 60012790921296 respectively) in a negligable amount of time (less than .01 in each of the first two examples) so we don't list this in the chart to save space. This table lists the exact (up to computer precision error) multiplicity as computed by MACAULAY2 or VINCI and the POLYPROB

(PP) results for different values of N. The entries in the "PP result" column are an average of 10 trials. Averaging trials appears to give slightly more accurate results. Last we include the CPU run times for each of the calculations.

Ex.	M2 result	VINCI result	PP result	N	# ineq.	PP time	M2 time	VINCI time
2.1	342	342	342.202	5000	5	.05	.25	.07
			342.04	10000		.07		
			342.043	20000		.10		
			342.093	50000		.18		
2.2	546	546	547.688	10000	14	.07	1.37	.07
			547.642	20000		.10		
			545.946	50000		.18		
			545.553	100000		.31		
2.3	-	$(6.001279)(10^{13})$	$(5.95065)(10^{13})$	10000	494	.73	-	.13
			$(5.98462)(10^{13})$	20000		1.42		
			$(6.0257)(10^{13})$	50000		3.47		
			$(6.01912)(10^{13})$	100000		6.98		
			$(6.01127)(10^{13})$	500000		34.93		
			$(6.00291)(10^{13})$	1000000		1:10.57		

At this point NORMALIZ seems to outperform all of the programs on these examples. POLYPROB is clearly much better than M2 on even medium problems. In terms of time VINCI appears to be the best of those programs in the chart. However, we note that POLYPROB will work as accurately if we give it an ideal of the form $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}, f_1, \ldots f_n)$ where f_i for at least one i is in the integral closure of the ideal $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n})$, but VINCI will fail to give the correct multiplicity in this case and as noted before, VINCI is only written using standard floating point arithmetic.

POLYPROB Implementation

The fundamental operation of our POLYPROB algorithm is a random trial: that is, generating a random vector within the simplex containing the polytope, and testing whether the vector is in the polytope. Thus, POLYPROB requires an efficient way to get random vectors uniformly distributed over a simplex. To see how to do this, first consider the general problem of generating a vector $(x_1, \ldots, x_n)$ uniformly distributed over an n–dimensional polytope $\mathcal{P}$. Given a description of the polytope, say as the convex hull of a set of vertices, we can calculate the minimum and maximum values for each coordinate of a vector in the polytope. That is, we can determine that the polytope lies within the hypercube $\prod_{i=1}^{n}[a_i, b_i]$. Our first task, then, is to pick $x_1 \in [a_1, b_1]$ according to an appropriate probability distribution.

Thus, for any $c \in [a_1, b_1]$ we can calculate $f(c) = \Pr(x_1 \in [a_1, c])$ by calculating the volume of $\mathcal{P} \cap \{(x_1, \ldots, x_n) : a_1 \leq x_1 \leq c\}$ as a percentage of the volume of $\mathcal{P}$. This gives us a monotone increasing distribution function $f : [a_1, b_1] \to [0, 1]$. It is from this distribution function that we want to sample x_1. If we can pick a random real number X uniformly distributed over $[0, 1]$, then we can just take $x_1 = f^{-1}[X]$. Once x_1 has been sampled, its value determines an $(n-1)$-dimensional cross-section of $\mathcal{P}$ so we have now reduced the problem to picking a smaller random vector $(x_2, \ldots, x_n)$ uniformly distributed over that cross-section. Thus we can iteratively pick $x_2, \ldots, x_n$ by the same algorithm used to pick x_1.

For general polytopes, there are very large practical problems with this algorithm. However, for simplices all of these problems disappear. Since Δ is a simplex we only need to perform this algorithm for simplices. Consider a simplex with one vertex at the origin and vertices $v_1, \ldots, v_n$ where $v_i = (0, \ldots, a_i, 0, \ldots, 0)$. Then

$$\Pr(x_1 \in [c, a_1]) = \left(\frac{a_1 - c}{a_1}\right)^n$$

so the inverse of the distribution function is just $f^{-1}(X) = 1 - X^{1/n}$ times the scaling factor a_i. And if we sample x_1, the cross-section of the simplex at x_1 is just the $(n-1)$-dimensional simplex with vertices at $(x_1, 0, 0, \ldots, 0)$ and $v_2 \ldots v_n$ where $v_i = (x_1, 0, \ldots, (1 - \frac{x_1}{a_1})a_i, \ldots)$.

The source code for our implementation of POLYPROB illustrates our application of this method; it is available at `http://www.math.rutgers.edu/~nweining/polyprob.tar.gz`.

3 Membership test for integral closure of monomial ideals

In this section we provide a linear programming solution to the membership test '$f \in \overline{I}$?'

Monomial ideals of finite co-length

We will provide now membership & completeness tests and a construction of the integral closure of monomial ideals of finite co-length. Our treatment is a by-product of the half-spaces description of the convex hull given in Eq. (3). We point out how the following oracle gives a solution to the membership and completeness tests and the construction task in case of an ideal of finite co-length.

Proposition 3.1 *Let I be a monomial ideal of finite co-length as above, and let f be a monomial. Denote by $\mathbf{e} = (e_1, \ldots, e_n)$ the exponent vector of f, by $\mathbf{v} = (v_1, \ldots, v_n)$, A and $\mathbf{b}$ the vectors and matrices associated to I as discussed above. Then f is integral over I if one of the two conditions holds:*

$$A \cdot \mathbf{e} \leq \mathbf{b},$$

$$\textstyle\sum_{i=1}^{n} \frac{e_i}{v_i} \geq 1.$$

Proof. These conditions simply express the fact that either $\mathbf{e}$ lies in the convex hull of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (in which case f would lie in the integral closure of $(\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_n})$), or that adding f to I does not affect the volume of $\mathcal{P}$. In the second case, $e(I) = e(I, f)$, f is integral over I by Rees' theorem. $\square$

Definition 3.2 A *membership oracle* for the integral closure of an ideal I is a boolean function $\mathcal{A}$ such that $f \in \overline{I}$ if and only if $\mathcal{A}(f) = \text{true}$.

Proposition 3.1 above shows that monomial ideals of finite co-length admit such oracles. We show now how given any membership oracle $\mathcal{A}$ for a monomial ideal I of finite co-length leads also to a completeness test. We begin with a general observation that shows some of the opportunities and difficulties in developing such tests.

Proposition 3.3 *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of finite co-length. Denote by $L = I\colon \mathfrak{m}$ the socle ideal of I. Then I is complete if and only if no element of $L \setminus I$ is integral over I.*

Proof. If $f \in \overline{I} \setminus I$, then for some power of $\mathfrak{m}$, $\mathfrak{m}^r f$ will contain non–trivial elements in the socle of I. The converse is clear. $\square$

Proposition 3.4 *Let I be a monomial ideal in $k[x_1, \ldots, x_d]$ of finite co-length and let $\mathcal{A}$ be a membership oracle for the integral closure of I. Let $\{f_1, \ldots, f_s\}$ be the monomials in $I : (x_1, \ldots, x_d) \setminus I$. Then*

$$I = \overline{I} \Longleftrightarrow \mathcal{A}(f_i) = \text{false}, \;\; i = 1, \ldots, s.$$

Proof. First, we consider the reverse direction. Let $L = I\colon (x_1, \ldots, x_d)$ be the socle ideal of I. L is generated by the f_i and monomials in I. Since $\overline{I}$ is a monomial ideal, if f is a monomial $\in \overline{I} \setminus I$, by multiplying by another monomial g, we obtain gf generating a nonzero element in the vector space L/I. This means that gf must be one of the f_i. Since gf is also integral over I, the assertion follows. The other assertion is obvious. $\square$

The construction of $\overline{I}$ follows in a straightforward manner:

If $I \neq \overline{I}$, define

$$I_1 = (I, \mathcal{A}(f_i) = \text{true}, \;\; i = 1, \ldots, s).$$

Since $\overline{I_1} = \overline{I}$, $\mathcal{A}$ is still a membership oracle for the integral closure of I_1 and we can repeat until $I_n = \overline{I}$. The program terminates by Proposition 3.4 and is has been implemented in MACAULAY2.

General monomial ideals

A more comprehensive membership test for the question "$f \in \overline{I}$", valid for any monomial ideal, is the following. However, this test lacks the effectivity of the method in the previous section.

Proposition 3.5 *Let $v_1, \ldots, v_m$ be a set of vectors in $\mathbb{N}^n$ and let A be the $n \times m$ matrix whose columns are the vectors $v_1, \ldots, v_m$. If $I = (\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_m})$, then a monomial $\mathbf{x}^b$ lies in the integral closure of I if and only if the linear program:*

$$\text{Maximize } x_1 + \cdots + x_m \qquad (*)$$

$$\text{Subject to } Ax \leq b \text{ and } x \geq 0$$

has an optimal value greater or equal than 1, which is attained at a vertex of the rational polytope$P = \{x \in \mathbb{R}^m \mid Ax \leq b$ and $x \geq 0\}$.

Proof. $\Rightarrow$) Let $\mathbf{x}^b \in \overline{I}$, that is, $\mathbf{x}^{pb} \in I^p$ for some positive integer p. There are non-negative integers r_i satisfying

$$\mathbf{x}^{pb} = \mathbf{x}^{\delta} (\mathbf{x}^{v_1})^{r_1} \cdots (\mathbf{x}^{v_m})^{r_m} \text{ and } r_1 + \cdots + r_m = p.$$

Hence the column vector c with entries $c_i = r_i/p$ satisfies

$$Ac \leq b \text{ and } c_1 + \cdots + c_m = 1.$$

This means that the linear program has an optimal value greater or equal than 1.

$\Leftarrow$) Observe that the vertices of P have rational entries (see [6, Theorem 18.1]) and that the maximum of $x_1 + \cdots + x_m$ is attained at a vertex of the polytope$P$, thus there are non-negative rational numbers $c_1, \ldots, c_m$ such that

$$c_1 + \cdots + c_m \geq 1 \text{ and } c_1 v_1 + \cdots + c_m v_m \leq b.$$

By induction on m it follows rapidly that there are rational numbers $\epsilon_1, \ldots, \epsilon_m$ such that

$$0 \leq \epsilon_i \leq c_i \ \forall i \text{ and } \sum_{i=1}^{m} \epsilon_i = 1.$$

Therefore there is a vector $\delta \in \mathbb{Q}^n$ with non-negative entries satisfying

$$b = \delta + \epsilon_1 v_1 + \cdots + \epsilon_m v_m.$$

Thus there is an integer $p > 0$ such that

$$pb = \underbrace{p\delta}_{\in \mathbb{N}^n} + \underbrace{p\epsilon_1}_{\in \mathbb{N}} v_1 + \cdots + \underbrace{p\epsilon_m}_{\in \mathbb{N}} v_m,$$

and consequently $\mathbf{x}^b \in \overline{I}$. $\square$

Remark 3.6 According to [6, Theorem 5.1] if the primal problem $(*)$ has an optimal solution x, then the dual problem

$$\text{Minimize } b_1 y_1 + \cdots + b_n y_n$$

Subject to $yA \geq \mathbf{1}$ and $y \geq 0$

has an optimal solution y such that the optimal values of the two problems coincide. Thus one can also use the dual problem to test whether x^b is in $\overline{I}$. Here $\mathbf{1}$ denotes the vector with all its entries equal to 1. The advantage of considering the dual is that one has a fixed polyhedron

$$Q = \{y \in \mathbb{R}^n \,|\, yA \geq \mathbf{1} \text{ and } y \geq 0\}$$

that can be used to test membership of any monomial $\mathbf{x}^b$, while in the primal problem the polytope P depends on b. Using *PORTA* one can readily obtain the vertices of the polyhedral set Q. The matrix M whose rows are the vertices of Q is a "membership test matrix" in the sense that a monomial $\mathbf{x}^b$ lies in $\overline{I}$ iff $Mb \geq \mathbf{1}$.

Let us illustrate the criterion with a previous example.

Example 3.7 Consider the ideal I of Example 1.4. To verify that $\mathbf{x}^b = x_1^2 \cdots x_8^2$ is in $\overline{I^3}$ one uses the following procedure in *Mathematica*

```
ieq:={
3x1 + 3x2 + 3x3 + 3x4 + 3x5<=2,
3x1 + 3x2 + 3x3 + 3x4 + 3x6<=2,
3x1 + 3x2 + 3x3 + 3x4 + 3x7<=2,
3x1 + 3x2 + 3x3 + 3x4 + 3x8<=2,
3x1 + 3x5 + 3x6 + 3x7 + 3x8<=2,
3x2 + 3x5 + 3x6 + 3x7 + 3x8<=2,
3x3 + 3x5 + 3x6 + 3x7 + 3x8<=2,
3x4 + 3x5 + 3x6 + 3x7 + 3x8<=2}

vars:={x1,x2,x3,x4,x5,x6,x7,x8}

f:=x1+x2+x3+x4+x5+x6+x7+x8

ConstrainedMax[f,ieq,vars]
```

The answer is:

```
{16/15,
{x1 -> 2/15, x2 -> 2/15, x3 -> 2/15, x4 -> 2/15, x5 -> 2/15,
   x6 -> 2/15, x7 -> 2/15, x8 -> 2/15}}
```

where the first entry is the optimal value and the other entries correspond to a vertex of the polytopes P. Using the criterion and the procedure above one rapidly verifies that $\mathbf{x}^b$ is a minimal generator of $\overline{I^3}$.

4 Computation of general multiplicities

We will make general observations about the computation of the multiplicity of arbitrary primary ideals. The input data is usually the following. Let $A = k[x_1, \ldots, x_r]/L$ be an affine algebra and let I be a primary ideal for some maximal ideal $\mathfrak{M}$ of A. The Hilbert–Samuel polynomial is the function, $n \gg 0$

$$n \mapsto \lambda(A/I^n) = \frac{e(I)}{d!}n^d + \text{lower terms}, \quad \dim A_{\mathfrak{M}} = d.$$

In other words, $e(I)$ is the ordinary multiplicity of the standard graded algebra

$$\mathrm{gr}_I(A) = \sum_{n \geq 0} I^n/I^{n+1}.$$

For the actual computation, ordinarily one needs a presentation of this algebra

$$\mathrm{gr}_I(A) = k[T_1, \ldots, T_m]/H,$$

where the right side is not always a standard graded algebra. In the special case of $I = (x_1, \ldots, x_r)A$ and L is a homogeneous ideal, one has that

$$\mathrm{gr}_M(A) \simeq A,$$

and therefore it can be computed in almost all computer algebra systems by making use of:

Theorem 4.1 (Macaulay Theorem) *Given an ideal I and a term ordering $>$, the mapping*

$$\text{NormalForm}\colon R/I \longrightarrow R/in_>(I) \tag{5}$$

is an isomorphism of k-vector spaces. If I is a homogeneous ideal and $>$ is a degree term ordering, then NormalForm *is an isomorphism of graded k–vector spaces, in particular the two rings have the same Hilbert function.*

For our case, this implies that

$$e(I) = \deg(A) = \deg(k[x_1, \ldots, x_r]/\mathrm{in}_>(L)),$$

where $>$ is any degree term ordering of the ring of polynomials $k[x_1, \ldots, x_r]$. We can turn the general problem into this case by the following observation (which hides the difficulties of the conversion). Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an $\mathfrak{m}$–primary ideal. To calculate the multiplicity $e(I)$ we need some form of access to a presentation of the associated graded ring $\mathrm{gr}_I(R)$,

$$\mathrm{gr}_I(R) = k[T_1, \ldots, T_s]/(f_1, \ldots, f_m),$$

in order to avail ourselves of the programs that determine Hilbert functions. A proposed solution, that uses heavily, Gröbner basis theory, is given in [14].

Alternatively, one can turn to indirect means. For instance, suppose $R = k[x_1, \ldots, x_d]$ is a ring of polynomials and I is an $(x_1, \ldots, x_d)$–primary ideal. Let J be a minimal reduction of I, then

$$e(I) = \lambda(R/J).$$

(A similar approach works whenever R is a Cohen–Macaulay ring.) If $>$ is a term order of R, then

$$\lambda(R/I) = \lambda(R/\mathrm{in}_>(J)).$$

The difficulty is to obtain J. It usually arises by taking a set of d generic linear combination of a generating system of I. In addition, even when I is homogeneous, J will not be homogeneous (often it is forbidden to be). One positive observation that can be made is:

Proposition 4.2 *Let I be an $(x_1, \ldots, x_d)$–primary ideal. For any term order $>$ of R,*

$$e(I) \leq e(in_>(I)) \leq d! \cdot e(I). \tag{6}$$

Proof. Denote $L = \mathrm{in}_>(I)$. The multiplicities are read from the leading coefficients of the Hilbert polynomials $\lambda(R/I^n)$ and $\lambda(R/L^n)$, $n \gg 0$. We note however that while $\lambda(R/I) = \lambda(R/L)$, for large n we can only guarantee

$$\lambda(R/I^n) = \lambda(R/\mathrm{in}_>(I^n)) \leq \lambda(R/L^n),$$

since the inclusion

$$(\mathrm{in}_>(I))^n \subset \mathrm{in}_>(I^n)$$

may be proper.

The other inequality will follow from Lech's formula ([13]) applied to the ideal L:

$$e(L) \leq d!\lambda(R/L)e(R) \leq d!e(I),$$

since $e(R) = 1$ and $\lambda(R/L) = \lambda(R/I) \leq e(I)$. □

As an illustration, let $I = (xy, x^2 + y^2) \subset k[x, y]$. Picking the deglex ordering with $x > y$, gives $L = \mathrm{in}_>(I) = (xy, x^2, y^3)$. We thus have

$$4 = e(I) < e(L) = 5.$$

We are now going to explain the equality $e(I) = e(L)$. Set $L_n = \mathrm{in}_>(I^n)$. Note that $B = \sum_{n \geq 0} L_n t^n$ is the Rees algebra of the filtration defined by L_n's. Actually, B is the initial algebra $\mathrm{in}_>(R[It])$ of the Rees algebra $R[It]$ for the extended term order of $R[t]$:

$$ft^r > gt^s \Leftrightarrow r > s \quad \text{or} \quad r = s \quad \text{and} \quad f > g.$$

In general, B is not Noetherian (which is the case in the simple example above, according to [9]).

Theorem 4.3 *Let I be an $(x_1, \ldots, x_d)$-primary ideal of the polynomial ring $k[x_1, \ldots, x_d]$, and let $>$ be a term ordering. The following conditions are equivalent:*

(a) *$e(I) = e(in_>(I))$.*

(b) *B is integral over $R[Lt]$, in particular B is Noetherian.*

Proof. (a) $\Rightarrow$ (b): To prove that B is contained in the integral closure of $R[Lt]$ it will be enough to show that for each s, the algebra $R[L_s t]$ is integral over $R[L^s t]$, in other words, to prove the assertion (b) for corresponding Veronese subalgebras.

Since, by hypothesis, the functions $\lambda(R/L^n)$ and $\lambda(R/I^n) = \lambda(R/L_n)$, for $n \gg 0$, are polynomials of degree d with the same leading coefficients, and we have

$$\lambda(R/(L^s)^n) \geq \lambda(R/L_s^n) \geq \lambda(R/L_{sn}) = \lambda(R/I^{sn}) = \lambda(R/(I^s)^n),$$

and

$$e(L^s) = s^d e(L) = s^d e(I) = e(I^s),$$

it follows that L^s and L_s have the same multiplicities. By Rees theorem ([15]), L_s is integral over L^s.

(b) $\Rightarrow$ (a): It is immediate. □

Some of these facts can be extended to more general affine algebras. Suppose I is a monomial ideal of finite co-length and $L \subset I$ is a monomial subideal. The multiplicity of I/L arises from the function

$$n \mapsto \lambda(R/(I^n + L)).$$

We will argue that there is a 'volume formula', similar to Proposition 1.5 that holds in this case. It is an application of the associativity formula for multiplicities: If $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal prime ideals of L of dimension $s = \text{height } L$, then

$$e(I/L) = \sum_{i=1}^{r} \lambda((R/L)_{\mathfrak{p}_i}) \cdot e((I + \mathfrak{p}_i)/\mathfrak{p}_i).$$

Once the $\mathfrak{p}_i$ have been found, we may apply Proposition 1.5 to each monomial ideal $I_i = (I + \mathfrak{p}_i)/\mathfrak{p}_i$. The other terms are co-lengths of monomial ideals. Indeed, the length l_i of the localization $(R/L)_{\mathfrak{p}_i}$ is obtained by setting to 1 in R and in L all the variables which do not belong to $\mathfrak{p}_i$. On the other hand, the ideal $I/\mathfrak{p}_i$ is obtained by setting to 0 the variables that lie in $\mathfrak{p}_i$.

Proposition 4.4 *The multiplicity of the 'monomial' ideal I/L is given by*

$$e(I/L) = \sum_{i=1}^{r} l_i \cdot e(I_i).$$

We can also make comparisons between multiplicities of ideals in general affine rings and the monomial case. Consider an ideal

$$I/L \subset A = R/L = k[x_1, \ldots, x_n]/L$$

of codimension d. For some term order, let L' and I' be the corresponding initial ideals. Denoting by $(\cdot)'$ the initial ideal operation, we have

$$\frac{I'^n + L'}{L'} \subset \frac{(I^n + L)'}{L'}, \;\; n \geq 0.$$

As in the case when $L = (0)$, we have

$$\lambda(R/(I^n + L)) = \lambda(R/(I^n + L)') \leq \lambda(R/(I'^n + L')), \;\; n \geq 0,$$

and consequently,

$$e(I/L) \leq e(I'/L').$$

On the other hand, by Lech's formula ([13]),

$$e(I'/L') \leq d! \cdot \lambda(R/I') \cdot e(R/L') = d! \cdot \lambda(R/I) \cdot e(R/L),$$

the substitution $e(R/L') = e(R/L)$ by Macaulay's theorem.

References

[1] D. Bayer and M. Stillman, *Macaulay*: A system for computation in algebraic geometry and commutative algebra, 1992. Available at `http://www.math.harvard.edu`.

[2] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge University Press, 1993.

[3] W. Bruns and R. Koch, *Normaliz*: A program to compute normalizations of semigroups. Available by anonymous `ftp from ftp.mathematik.Uni-Osnabrueck.DE/pub/osm/kommalg/software/`

[4] B. Büeler, A. Enge and K. Fukuda, Exact volume computations for convex polytopes: a practical study. In G. Kalai and G. Ziegler, editors, *Polytopes-Combinatorics and Computation*, DMV-Seminar **29**, Birkhäuser Verlag, 2000.

[5] T. Christof, revised by A. Löbel and M. Stoer, *PORTA*, 1997. Available at `http://www.zib.de/Optimization/Software/Porta`.

[6] V. Chvátal, *Linear Programming*, W. H. Freeman and Company, New York, 2000.

[7] A. Corso, C. Huneke and W. V. Vasconcelos, On the integral closure of ideals, *Manuscripta Math.* **95** (1998), 331–347.

[8] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Springer, Berlin Heidelberg New York, 1995.

[9] S. Eliahou, A problem about polynomial ideals, *Contemporary Math.* **58** (1986), 107–120.

[10] W. Feller, *An Introduction to Probability Theory and its Applications*, 3rd Edition, Wiley, New York, 1967.

[11] *GMP Libraries*, 1997. Available at `http://www.swox.com/gmp/`.

[12] D. Grayson and M. Stillman, Macaulay 2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2`.

[13] C. Lech, Note on multiplicities of ideals, *Arkiv för Matematik* **4** (1960), 63–86.

[14] T. Mora and M. E. Rossi, An algorithm for the Hilbert function of a primary ideal, *Comm. Algebra* **23** (1995), 1899–1911.

[15] D. Rees, α–transforms of local rings and a theorem on multiplicities of ideals, *Math. Proc. Camb. Phil. Soc.* **57** (1961), 8–17.

[16] L. Robbiano and M. Sweedler, Subalgebra bases, *in Commutative Algebra*, Proceedings, Salvador 1988 (W. Bruns and A. Simis, Eds.), Lecture Notes in Mathematics **1430**, Springer, Berlin Heidelberg New York, 1990, 61–87.

[17] Yu. A. Shreider, *The Monte Carlo Method*, Pergamon Press, Oxford, 1966.

[18] R. Stanley, *Enumerative Combinatorics*, Vol. I, Wadsworth & Brooks/Cole, Monterey, California, 1986.

[19] B. Teissier, Monômes, volumes et multiplicités, in *Introduction à la theorie des Singularitiés*, Vol. II (Lê Dung Trang, Ed.), Travaux en Cours **37**, Hermann, Paris, 1988.

[20] W. V. Vasconcelos, *Computational Methods in Commutative Algebra and Algebraic Geometry*, Springer, Heidelberg, 1998.

[21] R. Villarreal, *Monomial Algebras*, Monographs and Textbooks in Pure and Applied Mathematics **238**, Marcel Dekker, New York, 2001.

[22] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. II, Van Nostrand, Princeton, 1960.