Torsion freeness and normality of blowup rings of monomial ideals

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Abstract

Let $I = (x^{v_1}, \ldots, x^{v_q})$ be a monomial ideal of a polynomial ring $K[\mathbf{x}]$ over a field K and let A be the matrix with column vectors $v_1, \ldots, v_q$. We show that the associated graded ring of I is reduced if A is totally unimodular. Then we express the symbolic Rees algebra of I in terms of Hilbert bases provided that I is square-free. A normality criterion is presented which is adequate for monomial ideals.

1 Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let I be a monomial ideal of R generated by a finite set of monomials $\{x^{v_1}, \ldots, x^{v_q}\}$. As usual we use x^a as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n)$

is in $\mathbb{N}^n$. The three central objects of study here are the following blowup algebras: (a) the *Rees algebra*

$$R[It] := R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t],$$

where t is a new variable, (b) the *associated graded ring*

$$\mathrm{gr}_I(R) := R/I \oplus I/I^2 \oplus \cdots \oplus I^i/I^{i+1} \oplus \cdots \simeq R[It] \otimes_R (R/I),$$

with multiplication

$$(a + I^{i+1})(b + I^{j+1}) = ab + I^{i+j+1} \qquad (a \in I^i,\ b \in I^j),$$

and (c) the *symbolic Rees algebra*

$$R_s(I) := R + I^{(1)}t + I^{(2)}t^2 + \cdots + I^{(i)}t^i + \cdots \subset R[t],$$

where $I^{(i)}$ is the ith symbolic power of I.

It is well known [20, p. 168] that the integral closure of $R[It]$ in its field of fractions can be expressed as

$$\overline{R[It]} = R \oplus \overline{I}t \oplus \cdots \oplus \overline{I^i}t^i \oplus \cdots ,$$

where $\overline{I^i}$ is the integral closure of I^i. The integral closure of the ith power of I is again a monomial ideal [20, p. 169] given by:

$$\overline{I^i} = (\{x^a \in R \mid (x^a)^p \in I^{pi} \text{ for some } p \geq 1\}). \tag{1}$$

If $I^i = \overline{I^i}$ for $i \geq 1$, the ideal I is called *normal*. The ring $R[It]$ is said to be *normal* if $R[It]$ is equal to its integral closure. Thus $R[It]$ is normal if and only if I is normal. Recall that the ring $\mathrm{gr}_I(R)$ is called *reduced* if its nilradical is zero.

The content of the paper is as follows. Our first result proves that if A is the matrix with column vectors $v_1, \ldots, v_q$ and each $i \times i$ minor of A is 0 or ± 1 for all $i \geq 1$, then $\mathrm{gr}_I(R)$ is reduced (Theorem 2.3). As a consequence it is recovered that $R[It]$ is normal [2], see Remark 2.5.

We point out an effective criterion to determine when $\mathrm{gr}_I(R)$ is reduced (Proposition 3.4). If I is square-free, we give a description of its symbolic Rees algebra in terms of Hilbert bases (Theorem 3.5). As a consequence we recover that $R_s(I)$ is finitely generated [12].

In Section 4 we present a normality criterion which is adequate for monomial ideals (Theorem 4.4). To illustrate the practical use of this criterion it is shown that ideals of Veronese type are normal (Proposition 4.9). As a byproduct, using a result of [8], we derive that a monomial subring of Veronese type is the Ehrhart ring of a lattice polytope [7].

The main tools to show those results are commutative algebra and linear optimization techniques as described in [11, 19, 20, 23].

2 Torsion freeness of the associated graded ring

We begin with a basic result about the nilpotent elements of the associated graded ring.

LEMMA 2.1 *If I is a monomial ideal of R, then the nilradical of the associated graded ring of I is given by*

$$\mathrm{nil}(\mathrm{gr}_I(R)) \quad = \quad (\{\overline{x^\alpha} \in I^i/I^{i+1} |\, x^{s\alpha} \in I^{si+1}; i \geq 0; s \geq 1\}).$$

Proof: The nilradical of $\mathrm{gr}_I(R)$ is graded with respect to the fine grading, and thus it is generated by homogeneous elements. $\Box$

DEFINITION 2.2 An integral matrix A is called *totally unimodular* if each $i \times i$ minor of A is 0 or ± 1 for all $i \geq 1$.

THEOREM 2.3 *Let $I = (x^{v_1}, \ldots, x^{v_q})$ be a monomial ideal of R and let A be the $n \times q$ matrix with column vectors $v_1, \ldots, v_q$. If A is totally unimodular, then $\mathrm{gr}_I(R)$ is reduced.*

Proof: Let $m = x^\alpha$ be a monomial such that: (i) $\overline{m} \in I^i/I^{i+1}$ and (ii) $m^s \in I^{is+1}$, for some $0 \neq s \in \mathbb{N}$. By Lemma 2.1 it suffices to prove that $m \in I^{i+1}$. From (ii) there are $a_1, \ldots, a_q$ in $\mathbb{N}$ and $\delta \in \mathbb{N}^n$ such that

$$x^{s\alpha} = (x^{v_1})^{a_1} \cdots (x^{v_q})^{a_q} x^\delta \quad \text{and} \quad \textstyle\sum_{j=1}^q a_j = is + 1.$$

Hence the linear program

$$\begin{aligned} &\text{Maximize } f(y) = \textstyle\sum_{i=1}^q y_i \\ &\text{Subject to } Ay \leq \alpha \text{ and } y \geq 0 \end{aligned} \qquad (*)$$

has an optimal value greater or equal than $z_0 = i + 1/s$ because the rational vector $y_0 = (a_1/s, \ldots, a_q/s)$ satisfies $Ay_0 \leq \alpha$, $y_0 \geq 0$ and the sum of its entries is z_0. Note that the polyhedron $Q = \{y |\, Ay \leq \alpha; y \geq 0\}$ is bounded. Indeed any $y = (y_1, \ldots, y_q)$ in Q must satisfy $y_j \leq \max\{\alpha_1, \ldots, \alpha_q\}$ for all j, where α_j is the j*th* entry of α. By a result of Hoffman and Kruskal (see [11, Theorem 5.19 and Corollary 5.20]) the optimal value of the linear program $(*)$ is attained by an integral vector $b = (b_1, \ldots, b_q)$. Thus $b_1 + \cdots + b_q \geq i+1$. As $b \in Q$, we can write

$$x^\alpha = (x^{v_1})^{b_1} \cdots (x^{v_q})^{b_q} x^\gamma,$$

for some $\gamma \in \mathbb{N}^n$. This proves $x^\alpha \in I^{i+1}$, as required. $\Box$

DEFINITION 2.4 An ideal I of a ring R is called *normally torsion free* if $\mathrm{Ass}(R/I^i)$ is contained in $\mathrm{Ass}(R/I)$ for all $i \geq 1$ and $I \neq R$.

REMARK 2.5 Assume that I is a square-free monomial ideal of R. Then according to [10, Corollary 1.10] I is normally torsion free $\Leftrightarrow$ $\mathrm{gr}_I(R)$ is torsion-free over $R/I \Leftrightarrow \mathrm{gr}_I(R)$ is reduced. If $\mathrm{gr}_I(R)$ is reduced, then $R[It]$ is normal [1]. Hence it is seen that Theorem 2.3 is a generalization of both [2, Theorem 3.2] and [17, Theorem 5.9].

3 Rees cones and symbolic Rees algebras

Let $I = (x^{v_1}, \ldots, x^{v_q})$ be a square-free monomial ideal of $R = K[x_1, \ldots, x_n]$ of height $g \geq 2$ such that $\deg(x^{v_i}) \geq 2$ for all i and let

$$\mathcal{A}' = \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\} \subset \mathbb{R}^{n+1},$$

where e_i is the *ith* unit vector in $\mathbb{R}^{n+1}$.

The *Rees cone* of the point configuration $\mathcal{A} = \{v_1, \ldots, v_q\}$ is the rational polyhedral cone, denoted by $\mathbb{R}_+\mathcal{A}'$, consisting of the linear combinations of $\mathcal{A}'$ with non negative coefficients. Notice that $\dim(\mathbb{R}_+\mathcal{A}') = n+1$ and also that $H_{e_i} \cap \mathbb{R}_+\mathcal{A}'$ is a facet of $\mathbb{R}_+\mathcal{A}'$ for $i = 1, \ldots, n+1$. Thus according to [23] there is a unique irreducible representation

$$\mathbb{R}_+\mathcal{A}' = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap \cdots \cap H_{\ell_r}^+ \tag{2}$$

such that each ℓ_i has relatively prime integral entries. As usual H_a^+ denotes the closed halfspace

$$H_a^+ = \{\alpha \in \mathbb{R}^{n+1} \,|\, \langle \alpha, a \rangle \geq 0\}$$

and H_a is the hyperplane through the origin with normal vector a. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of I. For each $\mathfrak{p}_k$ set $C_k = \{x_i | x_i \in \mathfrak{p}_k\}$ and define the vector ℓ_k as:

$$\ell_k = -e_{n+1} + \sum_{x_i \in C_k} e_i \qquad (k = 1, \ldots, s). \tag{3}$$

This definition of ℓ_k is consistent with Eq. (2) because of the following lemma. In the sequel we assume that $\ell_1, \ldots, \ell_s$ are the vectors given by Eq. (3).

LEMMA 3.1 *$H_{\ell_1}^+, \ldots, H_{\ell_s}^+$ occur in the irreducible representation of* $\mathbb{R}_+\mathcal{A}'$.

Proof: Let $1 \leq k \leq s$. It suffices to verify that $F = \mathbb{R}_+\mathcal{A}' \cap H_{\ell_k}$ is a facet of the Rees cone. Note that $\mathbb{R}_+\mathcal{A}' \subset H^+_{\ell_k}$ and $\mathbb{R}_+\mathcal{A}' \not\subset H_{\ell_k}$ because $I \subset \mathfrak{p}_k$ and $\mathfrak{p}_k \subsetneq (x_1, \ldots, x_n)$. From the minimality of $\mathfrak{p}_k$ it follows that there is a linearly independent set $\mathcal{F} \subset \mathcal{A}' \cap H_{\ell_k}$ with $n = |\mathcal{F}|$. $\Box$

LEMMA 3.2 *If* $1 \leq j \leq r$ *and* $\langle \ell_j, e_{n+1} \rangle = -1$*, then* $1 \leq j \leq s$.

Proof: Let $\ell_j = (a_1, \ldots, a_n, -1)$. Since $\mathbb{R}_+\mathcal{A}' \subset H^+_{\ell_j}$ we get $a_i \geq 0$ for $i = 1, \ldots, n$. Consider the ideal $\mathfrak{p}$ of R generated by the set of x_i such that $a_i > 0$. Note that $I \subset \mathfrak{p}$ because $\langle \ell_j, (v_i, 1) \rangle \geq 0$ for all $1 \leq i \leq n$. For simplicity of notation assume that $\ell_j = (a_1, \ldots, a_m, 0, \ldots, 0, -1)$, where $a_i > 0$ for all i. Take an arbitrary vector $v \in \mathcal{A}'$ such that $v \in H_{\ell_j}$, then v satisfies the equation

$$a_1 x_1 + \cdots + a_m x_m = x_{n+1}.$$

Observe that v also satisfies the equation

$$x_1 + \cdots + x_m = x_{n+1}.$$

Using that H_{ℓ_j} contains n linearly independent vectors from $\mathcal{A}'$ we conclude that $H_{\ell_j} = H_b$, where $b = e_1 + \cdots + e_m - e_{n+1}$. Hence $\ell_j = b$ and consequently $a_i = 1$ for all i. It remains to show that $\mathfrak{p}$ is a minimal prime of I, and this follows from the irreducibility of Eq. (2). $\Box$

Next we present two results that highlight the importance of the irreducible representation of the Rees cone.

PROPOSITION 3.3 *If* $B = R[It]$ *is normal and* $\mathrm{Cl}(B)$ *is its divisor class group, then* $\mathrm{Cl}(B) \simeq \mathbb{Z}^r$ *and* $r \geq s$.

Proof: By [16, Theorem 1.1] the divisor class group of B is a free abelian group of finite rank. Using [5, Theorem 2] it is seen that r is the rank of the divisor class group of B. $\Box$

Since the program *Normaliz* [3] computes the irreducible representation of the Rees cone and the integral closure of $R[It]$ the following is an effective criterion for the reducedness of the associated graded ring.

PROPOSITION 3.4 (EFFECTIVE CRITERION) $\mathrm{gr}_I(R)$ *is reduced if and only if* $R[It]$ *is normal and* $\langle \ell_i, e_{n+1} \rangle = -1$ *for* $i = 1, \ldots, r$.

Proof: $\Rightarrow$) By [10, Theorem 1.11] and Proposition 3.3 we get that $R[It]$ is normal and $r = s$. Thus $\langle \ell_i, e_{n+1} \rangle = -1$ for all $i = 1, \ldots, r$.

$\Leftarrow$) Using Lemma 3.2 we obtain $r = s$. Therefore the associated graded ring is reduced by [10, Theorem 1.11]. $\Box$

Symbolic Rees algebra Next we introduce one of the main objects of this section. The *symbolic Rees cone* or *Simis cone* of the point configuration $\mathcal{A}$ is the rational polyhedral cone defined as:

$$\mathrm{Cn}(\mathcal{A}) := H_{e_1}^{+} \cap \cdots \cap H_{e_{n+1}}^{+} \cap H_{\ell_1}^{+} \cap \cdots \cap H_{\ell_s}^{+}. \tag{4}$$

By [11, Lemma 5.4] there exists a finite set $\mathcal{H} \subset \mathbb{N}^{n+1}$ such that

(a) $\mathrm{Cn}(\mathcal{A}) = \mathbb{R}_{+}\mathcal{H}$, and

(b) $\mathbb{Z}^{n+1} \cap \mathbb{R}_{+}\mathcal{H} = \mathbb{N}\mathcal{H}$,

where $\mathbb{N}\mathcal{H}$ is the additive subsemigroup of $\mathbb{N}^{n+1}$ generated by $\mathcal{H}$. The set $\mathcal{H}$ is called an integral *Hilbert basis* of $\mathrm{Cn}(\mathcal{A})$. If we require $\mathcal{H}$ to be minimal (with respect to containment), then $\mathcal{H}$ is unique [15].

For use below recall that the *ith* symbolic power of the monomial ideal I is given by $I^{(i)} = \mathfrak{p}_1^i \cap \cdots \cap \mathfrak{p}_s^i$.

THEOREM 3.5 *If $K[\mathbb{N}\mathcal{H}]$ is the semigroup ring of $\mathbb{N}\mathcal{H}$, then*

$$R_s(I) = K[\mathbb{N}\mathcal{H}].$$

Proof: Recall that $K[\mathbb{N}\mathcal{H}] = K[\{x^a t^b |\, (a,b) \in \mathbb{N}\mathcal{H}\}]$. Take $x^a t^b \in R_s(I)$, that is, $x^a \in \mathfrak{p}_i^b$ for all i. Hence $\langle (a,b), \ell_i \rangle \geq 0$ for all i or equivalently $(a,b) \in \mathrm{Cn}(\mathcal{A})$. Thus $(a,b) \in \mathbb{N}\mathcal{H}$ and $x^a t^b \in K[\mathbb{N}\mathcal{H}]$. Conversely take $x^a t^b \in K[\mathbb{N}\mathcal{H}]$, then (a,b) is in $\mathrm{Cn}(\mathcal{A})$ and $\langle (a,b), \ell_i \rangle \geq 0$ for all i. Hence $x^a \in \mathfrak{p}_i^b$ for all i and $x^a \in I^{(b)}$, as required. $\square$

COROLLARY 3.6 ([12]) *$R_s(I)$ is a finitely generated K-algebra.*

Proof: It follows at once from Theorem 3.5. $\square$

REMARK 3.7 The Rees algebra of I can be written as

$$R[It] = K[\{x^a t^b |\, (a,b) \in \mathbb{N}\mathcal{A}'\}].$$

On the other hand according to [22, Theorem 7.2.28] one has

$$\overline{R[It]} = K[\{x^a t^b |\, (a,b) \in \mathbb{Z}^{n+1} \cap \mathbb{R}_{+}\mathcal{A}'\}].$$

Hence $R[It]$ is normal if and only if $\mathbb{N}\mathcal{A}' = \mathbb{Z}^{n+1} \cap \mathbb{R}_{+}\mathcal{A}'$.

COROLLARY 3.8 $\overline{R[It]} = R_s(I) \iff \langle \ell_i, e_{n+1} \rangle = -1$ *for* $i = 1, \ldots, r$.

Proof: The equality $\overline{R[It]} = R_s(I)$ holds if and only if

$$\mathbb{Z}^{n+1} \cap \mathbb{R}_{+}\mathcal{A}' = \mathbb{Z}^{n+1} \cap \mathrm{Cn}(\mathcal{A}).$$

Hence the asserted equality holds if and only if the Rees cone and the Simis cone are equal. By the irreducibility of the representations of both cones it follows that $\overline{R[It]} = R_s(I)$ if and only if $r = s$. To finish the proof note that $r = s$ if and only if $\langle \ell_i, e_{n+1} \rangle = -1$ for $i = 1, \ldots, r$. $\square$

Hilbert bases of Rees cones For the remaining of the section we do not assume that I is generated by square-free monomials. Here we will show a degree bound for the minimal integral Hilbert basis of $\mathbb{R}_+\mathcal{A}'$.

REMARK 3.9 Let a be a point of a set A in $\mathbb{R}^n$ and let x be a point in $\mathrm{aff}(A)$ not lying in A. If $\lambda_0 = \sup\{\lambda \in [0,1] \mid (1-\lambda)a + \lambda x \in A\}$, then $x_0 = (1-\lambda_0)a + \lambda_0 x$ is a relative boundary point of A.

A monomial $x^\alpha t^b$ in $\overline{R[It]}$ correspond to a vector (α, b) in $\mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$. Thus it is natural to assign degree b to $\beta = (\alpha, b)$ and write $\deg_{n+1}(\beta) = b$.

PROPOSITION 3.10 *Let $\mathbb{R}_+\mathcal{A}'$ be the Rees cone of $\mathcal{A}$ and let $\mathcal{B}$ be the its minimal integral Hilbert basis. If $\beta \in \mathcal{B}$, then $\deg_{n+1}(\beta) < n$.*

Proof: For convenience of notation we write $\beta = (\alpha, b)$, where α is a vector in $\mathbb{N}^n$ and $b = \deg_{n+1}(\beta)$ is in $\mathbb{N}$. Note that $\mathbb{R}_+\mathcal{A}' \cap \mathbb{Z}^{n+1}$ is equal to $\mathbb{Q}_+\mathcal{A}' \cap \mathbb{Z}^{n+1}$ and $\dim(\mathbb{R}_+\mathcal{A}') = n+1$. Thus, since $(\alpha, b) \in \mathbb{R}_+\mathcal{A}'$, by Carathéodory's theorem [9, Theorem 2.3, p. 10] we can write

$$(\alpha, b) = \lambda_1(v_{i_1}, 1) + \cdots + \lambda_r(v_{i_r}, 1) + \mu_1 e_{j_1} + \cdots + \mu_s e_{j_s} \quad (\lambda_i, \mu_k \in \mathbb{Q}_+), \ (*)$$

where $\{(v_{i_1}, 1), \ldots, (v_{i_r}, 1), e_{j_1}, \ldots, e_{j_s}\}$ is a linearly independent set of vectors contained in $\mathcal{A}'$. By the minimality of $\mathcal{B}$ it is seen that $0 \leq \lambda_i < 1$ and $0 \leq \mu_k < 1$ for all i, k. From Eq. $(*)$ we get

$$b = \lambda_1 + \cdots + \lambda_r < r \implies b \leq r - 1.$$

If $r \leq n$, then $b \leq n-1$. Thus from now on we may assume $r = n+1$ and Eq. $(*)$ takes the simpler form

$$(\alpha, b) = \lambda_1(v_{i_1}, 1) + \cdots + \lambda_{n+1}(v_{i_{n+1}}, 1).$$

Consider the cone C generated by $\mathcal{A}'' = \{(v_{i_1}, 1), \ldots, (v_{i_{n+1}}, 1)\}$. Since $-e_1$ is not in C, using Remark 3.9 we obtain a point

$$x_0 = (1-\lambda_0)(\alpha, b) + \lambda_0(-e_1) \quad (0 \leq \lambda_0 < 1)$$

in the relative boundary of C. According to [23, Theorem 3.2.1] the relative boundary of C is the union of its facets. Therefore using that any facet of C is an n-dimensional cone generated by a subset of $\mathcal{A}''$ (see [22, Proposition 7.2.21]) together with the minimality of $\mathcal{B}$ it follows that we can write the vector (α, b) as:

$$(\alpha, b) = \rho_0 e_1 + \rho_1(v_{j_1}, 1) + \cdots + \rho_n(v_{j_n}, 1) \qquad (0 \leq \rho_i < 1 \, \forall i),$$

and consequently $b \leq n-1$, as required. $\square$

As a consequence we obtain (cf. [4, Theorem 3.3]):

Corollary 3.11 *$\overline{R[It]}$ is generated by monomials of t-degree at most $n-1$.*

Proof: If $\mathcal{B}$ is the minimal integral Hilbert basis of the Rees cone, then $\overline{R[It]} = K[\mathbb{N}\mathcal{B}]$ and we may apply Proposition 3.10. $\Box$

4 Normality criteria

In this section we present a simple normality criterion which is suitable to study the normality of monomial ideals. Then, as an application, we show that ideals of Veronese type are normal.

To prove the normality of a monomial ideal I it suffices to show that the power I^i is integrally closed for $i = 1, \ldots, n-1$, where n is the number of variables [14], see [13] for a generalization of this fact. If I has finite co-length, then its integral closure can be constructed [6, Proposition 3.4].

Proposition 4.1 *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be a proper ideal of R. Then I is integrally closed if and only if $IR_{\mathfrak{p}}$ is integrally closed for any prime ideal $I \subset \mathfrak{p} \neq \mathfrak{m}$ and $\overline{I} \cap (I\colon \mathfrak{m}) = I$.*

Proof: It follows from the proof of Proposition 4.2. $\Box$

Proposition 4.2 *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be a proper ideal of R. Then I is normal if and only if*

(a) *$IR_{\mathfrak{p}}$ is normal for any prime ideal $I \subset \mathfrak{p} \neq \mathfrak{m}$, and*

(b) *$\overline{I^r} \cap (I^r\colon \mathfrak{m}) = I^r$ for all $r \geq 1$.*

Proof: $\Rightarrow$) That (a) is satisfied follows from the fact that integral closures and powers of ideals commute with localizations [22, Proposition 3.3.4]. Part (b) is clearly satisfied.

$\Leftarrow$) Assume $I^r \neq \overline{I^r}$ for some $r \geq 1$. Take $\mathfrak{p} \in \mathrm{Ass}_R(M)$, where $M = \overline{I^r}/I^r$. Since $\mathfrak{p}$ is in the support of M we get $M_{\mathfrak{p}} \neq (0)$, and consequently by (a) we must have $\mathfrak{p} = \mathfrak{m}$. Hence there is an embedding

$$R/\mathfrak{m} \hookrightarrow M \qquad (\overline{1} \mapsto \overline{x_0}),$$

where $x_0 \in \overline{I^r} \setminus I^r$ and $\mathfrak{m} = \mathrm{ann}(\overline{x_0})$. From (b) one concludes that $x_0 \in I^r$, a contradiction. Thus $I^r = \overline{I^r}$ for all $r \geq 1$. $\Box$

The following result asserts that a normal monomial ideal stays normal if we make a variable, say x_n, equal to 1 or 0.

Proposition 4.3 *Let $I = (x^{v_1}, \ldots, x^{v_q})$ be a normal ideal. If each x^{v_i} is written as $x_n^{a_i} g_i$, where g_i is a monomial in $R' = K[x_1, \ldots, x_{n-1}]$, then*

(a) *$J = (g_1, \ldots, g_q) \subset R'$ is a normal ideal, and*

(b) *$L = (\{g_i |\, a_i = 0\}) \subset R'$ is a normal ideal.*

Proof: (a) We will show that $J^r = \overline{J^r}$ for all $r \geq 1$. Take a monomial x^α of R' that belong to $\overline{J^r}$. Then $(x^\alpha)^p \in J^{rp}$ for some $p \geq 1$ and we can write

$$(x^\alpha)^p = g_1^{b_1} \cdots g_q^{b_q} x^\beta,$$

where $\sum_{i=1}^q b_i = rp$ and $x^\beta \in R'$. Multiplying both sides of the equation above by x_n^{sp} with $s = \sum_{i=1}^q a_i b_i$, we get

$$(x_n^s x^\alpha)^p = (x_n^{a_1} g_1)^{b_1} \cdots (x_n^{a_q} g_q)^{b_q} x^\delta \in I^{rp}.$$

Hence $x_n^s x^\alpha \in I^r$ and we can write

$$x_n^s x^\alpha = (x_n^{a_1} g_1)^{c_1} \cdots (x_n^{a_q} g_q)^{c_q} x^\gamma,$$

where $\sum_{i=1}^q c_i = r$. Evaluating the last equality at $x_n = 1$ gives that $x^\alpha \in J^r$, as required. Part (b) follows using similar arguments. $\Box$

Theorem 4.4 *Let $I = (x^{v_1}, \ldots, x^{v_q}) \subset R = K[x_1, \ldots, x_n]$ and let J_i be the ideal of R generated by the monomials obtained from $\{x^{v_1}, \ldots, x^{v_q}\}$ by making $x_i = 1$. Then I is normal if and only if*

(a) *J_i is normal for all i (resp. $I_\mathfrak{p}$ is normal for any prime ideal $\mathfrak{p} \neq \mathfrak{m}$),*

(b) *$\overline{I^r} \cap (I^r : \mathfrak{m}) = I^r$ for all $r \geq 1$, where $\mathfrak{m} = (x_1, \ldots, x_n)$.*

Proof: $\Rightarrow$) Condition (a) follows from Proposition 4.3 (resp. using that integral closures and powers of ideals commute with localizations). Part (b) is clearly satisfied.

$\Leftarrow$) To prove that the rth power of I is equal to $\overline{I^r}$ set $M = \overline{I^r}/I^r$. We proceed by contradiction assuming $I^r \subsetneq \overline{I^r}$. Take an associated prime ideal $\mathfrak{p}$ of M. There are embeddings

$$R/\mathfrak{p} \overset{\varphi}{\hookrightarrow} M \hookrightarrow R/I^r.$$

Thus $\mathfrak{p}$ is an associated prime of the monomial ideal I^r. As a consequence $\mathfrak{p}$ is generated by a subset of $\{x_1, \ldots, x_n\}$ (see [22, Proposition 5.1.3]) and $\mathfrak{p} \subset \mathfrak{m}$. Note $\mathfrak{p} = \mathfrak{m}$, otherwise if $x_i \notin \mathfrak{p}$ for some i, then by (a) $IR_\mathfrak{p} = J_i R_\mathfrak{p}$ is normal and $M_\mathfrak{p} = (0)$, a contradiction because $\mathfrak{p}$ is in the support of M. Thus $\mathfrak{p} = \mathfrak{m}$. If $\overline{x_0} = \varphi(\overline{1})$ is the image of $\overline{1}$ under the first embedding above, we get $x_0 \in \overline{I^r} \setminus I^r$ and $x_0 \in (I^r : \mathfrak{m})$, which contradicts (b). $\Box$

Normality of an ideal of Veronese type Let $R=K[x_1,\ldots,x_n]$ be a ring of polynomials over an arbitrary field K. Given a sequence of integers $(s_1,s_2,\ldots,s_n;d)$ such that $1\leq s_j\leq d\leq\sum_{i=1}^n s_i$ for all j, we define $\mathcal{A}$ as the set of partitions

$$\mathcal{A}=\{(a_1,\ldots,a_n)\in\mathbb{Z}^n|\, a_1+\cdots+a_n=d;\ 0\leq a_i\leq s_i\,\forall\, i\},$$

and F as the set of monomials

$$F=\{x^a|\, a\in\mathcal{A}\}=\{f_1,\ldots,f_q\}.$$

DEFINITION 4.5 The ideal $I=(F)\subset R$ is said to be of *Veronese type* with defining sequence $(s_1,\ldots,s_n;d)$.

The monomial subring $K[F]\subset R$ and its toric ideal have been studied in [18]. Here we will focus in the ideal $I=(F)$ and its Rees algebra, rather than on $K[F]$.

LEMMA 4.6 *Let $\mathfrak{p}\neq\mathfrak{m}$ be a prime ideal of R and let $x_i\notin\mathfrak{p}$. For $k\neq i$ define $s_k'=\min\{s_k,d-s_i\}$. If $I=(F)$ and I' is the ideal of the ring $R'=K[x_1,\ldots,\widehat{x}_i,\ldots,x_n]$ generated by all the monomials*

$$x_1^{a_1'}\cdots x_{i-1}^{a_{i-1}'}x_{i+1}^{a_{i+1}'}\cdots x_n^{a_n'}$$

of R' of degree $d-s_i$ such that $a_k'\leq s_k'$ for $k\neq i$, then $IR_\mathfrak{p}=I'R_\mathfrak{p}$.

Proof: For simplicity of notation we assume $i=1$. Let $A=R_{x_1}$ be the localization of R at the multiplicative set $\{x_1^k\}_{k\geq 0}$. Since $A_\mathfrak{p}=R_\mathfrak{p}$ it suffices to show the equality $I'A=IA$. If $d=s_1$, then $x_1^d\in I$ and $s_k'=0$ for $k\geq 2$. Thus in this case $I'A=A$ and $IA=A$. Assume $d>s_1$. Take a monomial

$$m=x_2^{a_2'}\cdots x_n^{a_n'}$$

of I' of degree $d-s_1$. As $a_2'+\cdots+a_n'=d-s_1$ and $a_k'\leq s_k'\leq s_k$ for $k\geq 2$, we have $x_1^{s_1}m\in I$ and $m\in IA$. Therefore $I'A\subset IA$. Conversely take a monomial $m=x_1^{a_1}\cdots x_n^{a_n}$ of degree d in I. If $a_k\geq d-s_1$ for some $k\geq 2$, then $s_k'=d-s_1$ and $x_k^{d-s_1}\in I'$. Thus $m\in I'A$. On the contrary if $a_k<d-s_1$ for all $k\geq 2$, then $a_k\leq s_k'$ for all $k\geq 2$. Note $a_2+\cdots+a_n=d-a_1\geq d-s_1$. It is seen that there are integers $a_2',\ldots,a_n'$ such that $a_k'\leq a_k$ for $k\geq 2$ and $a_2'+\cdots+a_n'=d-s_1$. As a consequence the monomial

$$x_2^{a_2'}\cdots x_n^{a_n'}$$

belongs to I' and $m\in I'A$. Hence $IA\subset I'A$. $\square$

Remark 4.7 If $d > s_i$, note that I' is an ideal of Veronese type with defining sequence $(s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_n; d - s_i)$. If $d = s_i$, then $I'R_{\mathfrak{p}} = IR_{\mathfrak{p}} = R_{\mathfrak{p}}$.

Definition 4.8 The *support* of a monomial $f = x_1^{a_1} \cdots x_n^{a_n}$, denoted by $\mathrm{supp}(f)$, is a subset of the set of variables given by

$$\mathrm{supp}(f) = \{x_i \mid a_i > 0\}.$$

Proposition 4.9 *If $I = (F) \subset R$ is an ideal of Veronese type, then I is normal.*

Proof: For simplicity we keep the notation introduced above. The proof is by induction on d. If $d = 1$ the result is clear. Indeed in this case I is the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ and $\mathrm{gr}_{\mathfrak{m}}(R)$ is reduced because the identity matrix is totally unimodular. Thus $R[\mathfrak{m}t]$ is normal. Note that $\mathrm{gr}_{\mathfrak{m}}(R)$ is a polynomial ring over the field K, but we do not need this fact.

Assume $d \geq 2$ and that the result holds for ideals of Veronese type generated by monomials of degree $< d$. It suffices to prove that conditions (a) and (b) of Theorem 4.4 are satisfied. Take any prime $\mathfrak{p} \neq \mathfrak{m}$ and pick $x_i \notin \mathfrak{p}$. By Lemma 4.6 we obtain that $I'R_{\mathfrak{p}} = IR_{\mathfrak{p}}$. Hence using induction hypothesis and Remark 4.7 we get that $IR_{\mathfrak{p}}$ is normal. Thus condition (a) holds. To verify condition (b) we proceed by contradiction assuming

$$\overline{I^r} \cap (I^r : \mathfrak{m}) \neq I^r \tag{5}$$

for some $r \geq 1$. This means we can choose a monomial f in $\overline{I^r} \setminus I^r$ such that $x_i f$ is in I^r for all i. Hence there are $f_i = x_1^{a_{i1}} \cdots x_n^{a_{in}}$, $i = 1, \ldots, r$, of degree d such that $a_{ij} \leq s_j$ for all i, j and satisfying the equality

$$x_1 f = f_1 f_2 \cdots f_r h,$$

where h is a monomial with $\deg(h) > 0$, because $f \in \overline{I^r}$. Note $x_1 \notin \mathrm{supp}(h)$ because $f \notin I^r$. Thus we may assume $x_1 \in \mathrm{supp}(f_1)$ and $h = x_2^{c_2} x_3^{c_3} \cdots x_p^{c_p}$, where $c_i > 0$ for all i. Observe that $x_\ell^{s_\ell}$ divides f_1 for all $2 \leq \ell \leq p$, otherwise we can write

$$f = ((f_1 x_\ell)/x_1) f_2 \cdots f_r (h/x_\ell)$$

to derive $f \in I^r$, a contradiction. Thus we can write

$$f_1 = x_1^{a_{11}} x_2^{s_2} \cdots x_p^{s_p} x_{p+1}^{a_{1(p+1)}} \cdots x_n^{a_{1n}} \qquad (a_{11} > 0).$$

If $r = 1$, then $\deg_{x_2}(f) \geq s_2 + 1$ and since $x_2 f \in I$ we obtain $x_2 f = g h'$, with g a monomial of degree d in I. By degree considerations it follows that x_2 divides h' and $f \in I$, a contradiction. From now on we assume $r \geq 2$.

Case (I): $x_\ell^{s_\ell}$ does not divides f_k for some $2 \leq \ell \leq p$ and for some $2 \leq k \leq r$. Hence for each $x_i \in \mathrm{supp}(f_k)$ with $i \neq \ell$, we have that $x_i^{s_i}$ divides f_1, for otherwise if $x_i^{s_i}$ does not divides f_1 we write

$$f = h_1 f_2 \cdots f_{k-1} h_k f_{k+1} \cdots f_r (h/x_\ell),$$

where $h_1 = (x_i f_1)/x_1 \in I$ and $h_k = (x_\ell f_k)/x_i \in I$. Therefore $f \in I^r$, a contradiction. Hence, since we have already seen that also $x_\ell^{s_\ell}$ divides f_1, we obtain

$$s_\ell + \sum_{x_i \in \mathrm{supp}(f_k)\setminus\{x_\ell\}} s_i \leq d = \deg(f_1)$$

and consequently

$$d - a_{k\ell} = \sum_{x_i \in \mathrm{supp}(f_k)\setminus\{x_\ell\}} a_{ki} \leq \sum_{x_i \in \mathrm{supp}(f_k)\setminus\{x_\ell\}} s_i \leq d - s_\ell.$$

Thus $a_{k\ell} \geq s_\ell$ and $x_\ell^{s_\ell}$ divides f_k, a contradiction.

Case (II): $x_\ell^{s_\ell}$ divides f_k for $2 \leq \ell \leq p$ and $2 \leq k \leq r$. Since we also have that $x_\ell^{s_\ell}$ divides f_1, in this case we conclude $\deg_{x_2}(f) \geq rs_2 + 1$. Recall that $x_2 f \in I^r$, which by degree considerations readily implies $f \in I^r$, a contradiction.

Altogether we see that in both cases the inequality (5) leads to a contradiction. Hence condition (b) holds. $\Box$

Corollary 4.10 ([7]) *Let $P \subset \mathbb{R}^n$ be the lattice polytope* $\mathrm{conv}(\mathcal{A})$ *and let*

$$A(P) = K[\{x^\alpha t^i | \alpha \in \mathbb{Z}^n \cap iP\}] \subset R[t]$$

be the Ehrhart ring of P, then $A(P) = K[Ft]$. In particular $K[Ft]$ is normal.

Proof: Since $R[Ft]$ is normal, from [8, Proposition 3.15] we get the equality $A(P) = K[Ft]$. As $A(P)$ is always normal, so is $K[Ft]$. $\Box$

Definition 4.11 Let $I \subset R$ be an ideal of Veronese type with defining sequence $(s_1, \ldots, s_n; d)$. I is called the *dth square-free Veronese ideal* (resp. *dth Veronese ideal*) of R if $s_i = 1$ (resp. $s_i = d$) for all i.

Corollary 4.12 ([21]) *If I is the ideal of R generated by the square-free monomials of degree d, then I is a normal ideal.*

Proof: Since I is the *dth* square-free Veronese ideal of R, it is of Veronese type. Hence the result follows from Theorem 4.9. $\Box$

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