A note on Rees algebras and the MFMC property

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Abstract

We study irreducible representations of Rees cones and characterize the max-flow min-cut property of clutters in terms of the normality of Rees algebras and the integrality of certain polyhedra. Then we present some applications to combinatorial optimization and commutative algebra. As a byproduct we obtain an effective method, based on the program *Normaliz* [4], to determine whether a given clutter satisfy the max-flow min-cut property. Let $\mathcal{C}$ be a clutter and let I be its edge ideal. We prove that $\mathcal{C}$ has the max-flow min-cut property if and only if I is normally torsion free, that is, $I^i = I^{(i)}$ for all $i \geq 1$, where $I^{(i)}$ is the *ith* symbolic power of I.

1 Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let $I \subset R$ be a monomial ideal minimally generated by $x^{v_1}, \ldots, x^{v_q}$. As usual we will use x^a as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. Consider the $n \times q$ matrix A with column vectors $v_1, \ldots, v_q$. A *clutter* with vertex set X is a family of subsets of X, called edges, none of which is included in another. A basic example of clutter is a graph. If A has entries in $\{0, 1\}$, then A defines in a natural way a *clutter* $\mathcal{C}$ by taking $X = \{x_1, \ldots, x_n\}$ as vertex set and $E = \{S_1, \ldots, S_q\}$ as edge set, where S_i is the support of x^{v_i}, i.e., the set of variables that occur in x^{v_i}. In this case we call I the *edge ideal* of the clutter $\mathcal{C}$ and write $I = I(\mathcal{C})$. Edge ideals are also called *facet ideals* [9]. This notion has been studied by Faridi [10] and Zheng [18]. The matrix A is often refer to as the *incidence* matrix of $\mathcal{C}$.

The *Rees algebra* of I is the R-subalgebra:

$$R[It] := R[\{x^{v_1}t, \ldots, x^{v_q}t\}] \subset R[t],$$

[1]Partially supported by CONACyT grants 49835-F, 49251-F and SNI.

where t is a new variable. In our situation $R[It]$ is also a K-subalgebra of $K[x_1,\ldots,x_n,t]$. The *Rees cone* of I is the rational polyhedral cone in $\mathbb{R}^{n+1}$, denoted by $\mathbb{R}_+\mathcal{A}'$, consisting of the non-negative linear combinations of the set

$$\mathcal{A}' := \{e_1,\ldots,e_n,(v_1,1),\ldots,(v_q,1)\} \subset \mathbb{R}^{n+1},$$

where e_i is the *ith* unit vector. Thus $\mathcal{A}'$ is the set of exponent vectors of the set of monomials $\{x_1,\ldots,x_n,x^{v_1}t,\ldots,x^{v_q}t\}$, that generate $R[It]$ as a K-algebra.

The first main result of this note (Theorem 3.2) shows that the irreducible representation of the Rees cone, as a finite intersection of closed half-spaces, can be expressed essentially in terms of the vertices of the *set covering polyhedron*:

$$Q(A) := \{x \in \mathbb{R}^n \mid x \geq 0,\ xA \geq \mathbf{1}\}.$$

Here $\mathbf{1} = (1,\ldots,1)$. The second main result (Theorem 3.4) is an algebro-combinatorial description of the max-flow min-cut property of the clutter $\mathcal{C}$ in terms of a purely algebraic property (the normality of $R[It]$) and an integer programming property (the integrality of the rational polyhedron $Q(A)$). Some applications will be shown. For instance we give an effective method, based on the program *Normaliz* [4], to determine whether a given clutter satisfy the max-flow min-cut property (Remark 3.5). We prove that $\mathcal{C}$ has the max-flow min-cut property if and only if $I^i = I^{(i)}$ for $i \geq 1$, where $I^{(i)}$ is the *ith* symbolic power of I (Corollary 3.14). There are other interesting links between algebraic properties of Rees algebras and combinatorial optimization problems of clutters [11].

Our main references for Rees algebras and combinatorial optimization are [3, 14] and [12] respectively.

2 Preliminaries

For convenience we quickly recall some basic results, terminology, and notation from polyhedral geometry.

A set $C \subset \mathbb{R}^n$ is a *polyhedral set* (resp. *cone*) if $C = \{x \mid Bx \leq b\}$ for some matrix B and some vector b (resp. $b = 0$). By the finite basis theorem [17, Theorem 4.1.1] a polyhedral cone $C \subsetneq \mathbb{R}^n$ has two representations:

Minkowski representation $C = \mathbb{R}_+\mathcal{B}$ with $\mathcal{B} = \{\beta_1,\ldots,\beta_r\}$ a finite set, and

Implicit representation $C = H^+_{c_1} \cap \cdots \cap H^+_{c_s}$ for some $c_1,\ldots,c_s \in \mathbb{R}^n \setminus \{0\}$,

where $\mathbb{R}_+$ is the set of non-negative real numbers, $\mathbb{R}_+\mathcal{B}$ is the cone generated by $\mathcal{B}$ consisting of the set of linear combinations of $\mathcal{B}$ with coefficients in $\mathbb{R}_+$, H_{c_i} is the hyperplane of $\mathbb{R}^n$ through the origin with normal vector c_i, and $H^+_{c_i} = \{x \mid \langle x, c_i\rangle \geq 0\}$ is the positive *closed half-space* bounded by H_{c_i}. Here $\langle\ ,\ \rangle$ denotes the usual inner product. These two representations satisfy the *duality theorem for cones*:

$$H_{\beta_1}^+ \cap \cdots \cap H_{\beta_r}^+ = \mathbb{R}_+ c_1 + \cdots + \mathbb{R}_+ c_s, \tag{1}$$

see [13, Corollary 7.1a] and its proof. The *dual cone* of C is defined as

$$C^* := \bigcap_{c \in C} H_c^+ = \bigcap_{a \in \mathcal{B}} H_a^+.$$

By the duality theorem $C^{**} = C$. An implicit representation of C is called *irreducible* if none of the closed half-spaces $H_{c_1}^+, \ldots, H_{c_s}^+$ can be omitted from the intersection. Note that the left hand side of Eq. (1) is an irreducible representation of C^* if and only if no proper subset of $\mathcal{B}$ generates C.

3 Rees cones, normality and the MFMC property

To avoid repetions, throughout the rest of this note we keep the notation and assumptions of Section 1.

Notice that the Rees cone $\mathbb{R}_+\mathcal{A}'$ has dimension $n+1$. A subset $F \subset \mathbb{R}^{n+1}$ is called a *facet* of $\mathbb{R}_+\mathcal{A}'$ if $F = \mathbb{R}_+\mathcal{A}' \cap H_a$ for some hyperplane H_a such that $\mathbb{R}_+\mathcal{A}' \subset H_a^+$ and $\dim(F) = n$. It is not hard to see that the set

$$F = \mathbb{R}_+\mathcal{A}' \cap H_{e_i} \quad (1 \leq i \leq n+1)$$

defines a facet of $\mathbb{R}_+\mathcal{A}'$ if and only if either $i = n+1$ or $1 \leq i \leq n$ and $\langle e_i, v_j \rangle = 0$ for some column v_j of A. Consider the index set

$$\mathcal{J} = \{1 \leq i \leq n | \, \langle e_i, v_j \rangle = 0 \text{ for some } j\} \cup \{n+1\}.$$

Using [17, Theorem 3.2.1] it is seen that the Rees cone has a unique irreducible representation

$$\mathbb{R}_+\mathcal{A}' = \left(\bigcap_{i \in \mathcal{J}} H_{e_i}^+ \right) \bigcap \left(\bigcap_{i=1}^{r} H_{a_i}^+ \right) \tag{2}$$

such that $0 \neq a_i \in \mathbb{Q}^{n+1}$ and $\langle a_i, e_{n+1} \rangle = -1$ for all i. A point x_0 is called a *vertex* or an *extreme point* of $Q(A)$ if $\{x_0\}$ is a proper face of $Q(A)$.

Lemma 3.1 *Let $a = (a_{i1}, \ldots, a_{iq})$ be the ith row of the matrix A and define $k = \min\{a_{ij} | \, 1 \leq j \leq q\}$. If $a_{ij} > 0$ for all j, then e_i/k is a vertex of $Q(A)$.*

Proof. Set $x_0 = e_i/k$. Clearly $x_0 \in Q(A)$ and $\langle x_0, v_j \rangle = 1$ for some j. Since $\langle x_0, e_\ell \rangle = 0$ for $\ell \neq i$, the point x_0 is a basic feasible solution of $Q(A)$. Then by [1, Theorem 2.3] x_0 is a vertex of $Q(A)$. □

Theorem 3.2 *Let V be the vertex set of $Q(A)$. Then*

$$\mathbb{R}_+\mathcal{A}' = \left(\bigcap_{i\in\mathcal{J}} H^+_{e_i}\right)\bigcap\left(\bigcap_{\alpha\in V} H^+_{(\alpha,-1)}\right)$$

is the irreducible representation of the Rees cone of I.

Proof. Let $V = \{\alpha_1, \ldots, \alpha_p\}$ be the set of vertices of $Q(A)$ and let

$$\mathcal{B} = \{e_i |\, i \in \mathcal{J}\} \cup \{(\alpha, -1)|\, \alpha \in V\}.$$

First we dualize Eq. (2) and use the duality theorem for cones to obtain

$$\begin{aligned}(\mathbb{R}_+\mathcal{A}')^* &= \{y \in \mathbb{R}^{n+1} |\, \langle y, x\rangle \geq 0, \forall\, x \in \mathbb{R}_+\mathcal{A}'\} \\ &= H^+_{e_1} \cap \cdots \cap H^+_{e_n} \cap H^+_{(v_1,1)} \cap \cdots \cap H^+_{(v_q,1)} \\ &= \sum_{i\in\mathcal{J}} \mathbb{R}_+e_i + \mathbb{R}_+a_1 + \cdots + \mathbb{R}_+a_r. \end{aligned} \quad (3)$$

Next we show the equality

$$(\mathbb{R}_+\mathcal{A}')^* = \mathbb{R}_+\mathcal{B}. \quad (4)$$

The right hand side is clearly contained in the left hand side because a vector α belongs to $Q(A)$ if and only if $(\alpha, -1)$ is in $(\mathbb{R}_+\mathcal{A}')^*$. To prove the reverse containment observe that by Eq. (3) it suffices to show that $a_k \in \mathbb{R}_+\mathcal{B}$ for all k. Writing $a_k = (c_k, -1)$ and using $a_k \in (\mathbb{R}_+\mathcal{A}')^*$ gives $c_k \in Q(A)$. The set covering polyhedron can be written as

$$Q(A) = \mathbb{R}_+e_1 + \cdots + \mathbb{R}_+e_n + \mathrm{conv}(V),$$

where $\mathrm{conv}(V)$ denotes the convex hull of V, this follows from the structure of polyhedra by noticing that the characteristic cone of $Q(A)$ is precisely $\mathbb{R}^n_+$ (see [13, Chapter 8]). Thus we can write

$$c_k = \lambda_1e_1 + \cdots + \lambda_ne_n + \mu_1\alpha_1 + \cdots + \mu_p\alpha_p,$$

where $\lambda_i \geq 0$, $\mu_j \geq 0$ for all i, j and $\mu_1 + \cdots + \mu_p = 1$. If $1 \leq i \leq n$ and $i \notin \mathcal{J}$, then the ith row of A has all its entries positive. Thus by Lemma 3.1 we get that e_i/k_i is a vertex of $Q(A)$ for some $k_i > 0$. To avoid cumbersome notation we denote e_i and $(e_i, 0)$ simply by e_i, from the context the meaning of e_i should be clear. Therefore from the equalities

$$\sum_{i\notin\mathcal{J}} \lambda_ie_i = \sum_{i\notin\mathcal{J}} \lambda_ik_i\left(\frac{e_i}{k_i}\right) = \sum_{i\notin\mathcal{J}} \lambda_ik_i\left(\frac{e_i}{k_i}, -1\right) + \left(\sum_{i\notin\mathcal{J}} \lambda_ik_i\right)e_{n+1}$$

we conclude that $\sum_{i\notin\mathcal{J}} \lambda_i e_i$ is in $\mathbb{R}_+\mathcal{B}$. From the identities

$$\begin{aligned} a_k &= (c_k, -1) = \lambda_1 e_1 + \cdots + \lambda_n e_n + \mu_1(\alpha_1, -1) + \cdots + \mu_p(\alpha_p, -1) \\ &= \sum_{i\notin\mathcal{J}} \lambda_i e_i + \sum_{i\in\mathcal{J}\setminus\{n+1\}} \lambda_i e_i + \sum_{i=1}^{p} \mu_i(\alpha_i, -1) \end{aligned}$$

we obtain that $a_k \in \mathbb{R}_+\mathcal{B}$, as required. Taking duals in Eq. (4) we get

$$\mathbb{R}_+\mathcal{A}' = \bigcap_{a\in\mathcal{B}} H_a^+. \tag{5}$$

Thus, by the comments at the end of Section 2, the proof reduces to showing that $\beta \notin \mathbb{R}_+(\mathcal{B}\setminus\{\beta\})$ for all $\beta \in \mathcal{B}$. To prove this we will assume that $\beta \in \mathbb{R}_+(\mathcal{B}\setminus\{\beta\})$ for some $\beta \in \mathcal{B}$ and derive a contradiction.

Case (I): $\beta = (\alpha_j, -1)$. For simplicity assume $\beta = (\alpha_p, -1)$. We can write

$$(\alpha_p, -1) = \sum_{i\in\mathcal{J}} \lambda_i e_i + \sum_{j=1}^{p-1} \mu_j(\alpha_j, -1), \qquad (\lambda_i \geq 0; \mu_j \geq 0).$$

Consequently

$$\alpha_p = \sum_{i\in\mathcal{J}\setminus\{n+1\}} \lambda_i e_i + \sum_{j=1}^{p-1} \mu_j \alpha_j \tag{6}$$

$$-1 = \lambda_{n+1} - (\mu_1 + \cdots + \mu_{p-1}). \tag{7}$$

To derive a contradiction we claim that $Q(A) = \mathbb{R}_+^n + \mathrm{conv}(\alpha_1, \ldots, \alpha_{p-1})$, which is impossible because by [2, Theorem 7.2] the vertices of $Q(A)$ would be contained in $\{\alpha_1, \ldots, \alpha_{p-1}\}$. To prove the claim note that the right hand side is clearly contained in the left hand side. For the other inclusion take $\gamma \in Q(A)$ and write

$$\begin{aligned} \gamma &= \sum_{i=1}^{n} b_i e_i + \sum_{i=1}^{p} c_i \alpha_i \qquad (b_i, c_i \geq 0; \sum_{i=1}^{p} c_i = 1) \\ &\overset{(6)}{=} \delta + \sum_{i=1}^{p-1} (c_i + c_p \mu_i)\alpha_i \qquad (\delta \in \mathbb{R}_+^n). \end{aligned}$$

Therefore using the inequality

$$\sum_{i=1}^{p-1} (c_i + c_p\mu_i) = \sum_{i=1}^{p-1} c_i + c_p \left(\sum_{i=1}^{p-1} \mu_i\right) \overset{(7)}{=} (1 - c_p) + c_p(1 + \lambda_{n+1}) \geq 1$$

we get $\gamma \in \mathbb{R}_+^n + \mathrm{conv}(\alpha_1, \ldots, \alpha_{p-1})$. This proves the claim.

Case (II): $\beta = e_k$ for some $k \in \mathcal{J}$. First we consider the subcase $k \leq n$. The subcase $k = n+1$ can be treated similarly. We can write

$$e_k = \sum_{i\in\mathcal{J}\setminus\{k\}}\lambda_i e_i + \sum_{i=1}^{p}\mu_i(\alpha_i, -1), \qquad (\lambda_i \geq 0; \mu_i \geq 0).$$

From this equality we get $e_k = \sum_{i=1}^{p}\mu_i\alpha_i$. Hence $e_k A \geq (\sum_{i=1}^{p}\mu_i)\mathbf{1} > 0$, a contradiction because $k \in \mathcal{J}$ and $\langle e_k, v_j\rangle = 0$ for some j. □

Clutters with the max-flow min-cut property For the rest of this section we assume that A is a $\{0,1\}$-matrix, i.e., I is a square-free monomial ideal.

Definition 3.3 The clutter $\mathcal{C}$ has the *max-flow min-cut* (MFMC) property if both sides of the LP-duality equation

$$\min\{\langle\alpha, x\rangle \,|\, x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1}\rangle \,|\, y \geq 0; Ay \leq \alpha\} \tag{8}$$

have integral optimum solutions x and y for each non-negative integral vector α.

It follows from [13, pp. 311-312] that $\mathcal{C}$ has the MFMC property if and only if the maximum in Eq. (8) has an optimal integral solution y for each non-negative integral vector α. In optimization terms [12] this means that the clutter $\mathcal{C}$ has the MFMC property if and only if the system of linear inequalities $x \geq 0$; $xA \geq \mathbf{1}$ that define $Q(A)$ is *totally dual integral* (TDI). The polyhedron $Q(A)$ is said to be *integral* if $Q(A)$ has only integral vertices.

Next we recall two descriptions of the integral closure of $R[It]$ that yield some formulations of the normality property of $R[It]$. Let $\mathbb{N}\mathcal{A}'$ be the subsemigroup of $\mathbb{N}^{n+1}$ generated by $\mathcal{A}'$, consisting of the linear combinations of $\mathcal{A}'$ with non-negative integer coefficients. The Rees algebra of the ideal I can be written as

$$\begin{aligned} R[It] &= K[\{x^a t^b \,|\, (a,b) \in \mathbb{N}\mathcal{A}'\}] &(9)\\ &= R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t]. &(10)\end{aligned}$$

According to [16, Theorem 7.2.28] and [15, p. 168] the integral closure of $R[It]$ in its field of fractions can be expressed as

$$\begin{aligned} \overline{R[It]} &= K[\{x^a t^b \,|\, (a,b) \in \mathbb{Z}\mathcal{A}' \cap \mathbb{R}_+\mathcal{A}'\}] &(11)\\ &= R \oplus \overline{I}t \oplus \cdots \oplus \overline{I^i}t^i \oplus \cdots, &(12)\end{aligned}$$

where $\overline{I^i} = (\{x^a \in R \,|\, \exists\, p \geq 1; (x^a)^p \in I^{pi}\})$ is the integral closure of I^i and $\mathbb{Z}\mathcal{A}'$ is the subgroup of $\mathbb{Z}^{n+1}$ generated by $\mathcal{A}'$. Notice that in our situation we have the equality $\mathbb{Z}\mathcal{A}' = \mathbb{Z}^{n+1}$. Hence, by Eqs. (9) to (12), we get that $R[It]$ is a normal domain if and only if any of the following two conditions hold: (a) $\mathbb{N}\mathcal{A}' = \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$, (b) $I^i = \overline{I^i}$ for $i \geq 1$.

Theorem 3.4 *The clutter $\mathcal{C}$ has the* MFMC *property if and only if $Q(A)$ is an integral polyhedron and $R[It]$ is a normal domain.*

Proof. $\Rightarrow$) By [13, Corollary 22.1c] the polyhedron $Q(A)$ is integral. Next we show that $R[It]$ is normal. Take $x^{\alpha}t^{\alpha_{n+1}} \in \overline{R[It]}$. Then $(\alpha, \alpha_{n+1}) \in \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$. Hence $Ay \leq \alpha$ and $\langle y, \mathbf{1}\rangle = \alpha_{n+1}$ for some vector $y \geq 0$. Therefore one concludes that the optimal value of the linear program

$$\max\{\langle y, \mathbf{1}\rangle \mid y \geq \mathbf{0};\ Ay \leq \alpha\}$$

is greater or equal than α_{n+1}. Since A has the MFMC property, this linear program has an optimal integral solution y_0. Thus there exists an integral vector y_0' such that

$$\mathbf{0} \leq y_0' \leq y_0 \ \text{ and } \ |y_0'| = \alpha_{n+1}.$$

Therefore

$$\begin{pmatrix} \alpha \\ \alpha_{n+1} \end{pmatrix} = \begin{pmatrix} A \\ \mathbf{1} \end{pmatrix} y_0' + \begin{pmatrix} A \\ \mathbf{0} \end{pmatrix} (y_0 - y_0') + \begin{pmatrix} \alpha \\ 0 \end{pmatrix} - \begin{pmatrix} A \\ \mathbf{0} \end{pmatrix} y_0$$

and $(\alpha, \alpha_{n+1}) \in \mathbb{N}\mathcal{A}'$. This proves that $x^{\alpha}t^{\alpha_{n+1}} \in R[It]$, as required.

$\Leftarrow$) Assume that A does not satisfy the MFMC property. There exists an $\alpha_0 \in \mathbb{N}^n$ such that if y_0 is an optimal solution of the linear program:

$$\max\{\langle y, \mathbf{1}\rangle \mid y \geq \mathbf{0};\ Ay \leq \alpha_0\}, \qquad (*)$$

then y_0 is not integral. We claim that also the optimal value $|y_0| = \langle y_0, \mathbf{1}\rangle$ of this linear program is not integral. If $|y_0|$ is integral, then $(\alpha_0, |y_0|)$ is in $\mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$. As $R[It]$ is normal, we get that $(\alpha_0, |y_0|)$ is in $\mathbb{N}\mathcal{A}'$, but this readily yields that the linear program $(*)$ has an integral optimal solution, a contradiction. This completes the proof of the claim.

Now, consider the dual linear program:

$$\min\{\langle x, \alpha_0\rangle \mid x \geq \mathbf{0},\ xA \geq \mathbf{1}\}.$$

By [17, Theorem 4.1.6]) the optimal value of this linear program is attained at a vertex x_0 of $Q(A)$. Then by the LP duality theorem [12, Theorem 3.16] we get $\langle x_0, \alpha_0\rangle = |y_0| \notin \mathbb{Z}$. Hence x_0 is not integral, a contradiction to the integrality of the set covering polyhedron $Q(A)$. $\Box$

Remark 3.5 The program *Normaliz* [4, 5] computes the irreducible representation of a Rees cone and the integral closure of $R[It]$. Thus one can effectively use Theorems 3.2 and 3.4 to determine whether a given clutter $\mathcal{C}$ as the max-flow min-cut property. See example below for a simple illustration.

Example 3.6 Let $I = (x_1x_5, x_2x_4, x_3x_4x_5, x_1x_2x_3)$. Using *Normaliz* [4] with the input file:

```
4
5
1 0 0 0 1
0 1 0 1 0
0 0 1 1 1
1 1 1 0 0
3
```

we get the output file:

```
9 generators of integral closure of Rees algebra:
  1  0  0  0  0  0
  0  1  0  0  0  0
  0  0  1  0  0  0
  0  0  0  1  0  0
  0  0  0  0  1  0
  1  0  0  0  1  1
  0  1  0  1  0  1
  0  0  1  1  1  1
  1  1  1  0  0  1

10 support hyperplanes:
   0   0   1   1   1  -1
   1   0   0   0   0   0
   0   1   0   0   0   0
   0   0   0   0   0   1
   0   0   1   0   0   0
   1   0   0   1   0  -1
   0   0   0   1   0   0
   0   0   0   0   1   0
   0   1   0   0   1  -1
   1   1   1   0   0  -1
```

The first block shows the exponent vectors of the generators of the integral closure of $R[It]$, thus $R[It]$ is normal. The second block shows the irreducible representation of the Rees cone of I, thus using Theorem 3.2 we obtain that $Q(A)$ is integral. Altogether Theorem 3.4 proves that the clutter $\mathcal{C}$ associated to I has the max-flow min-cut property.

Definition 3.7 A set $C \subset X$ is a *minimal vertex cover* of a clutter $\mathcal{C}$ if every edge of $\mathcal{C}$ contains at least one vertex in C and C is minimal w.r.t. this property. A set of edges of $\mathcal{C}$ is *independent* if no two of them have a common vertex. We denote by $\alpha_0(\mathcal{C})$ the smallest number of vertices in any minimal vertex cover of $\mathcal{C}$, and by $\beta_1(\mathcal{C})$ the maximum number of independent edges of $\mathcal{C}$.

Definition 3.8 Let $X = \{x_1, \ldots, x_n\}$ and let $X' = \{x_{i_1}, \ldots, x_{i_r}, x_{j_1}, \ldots, x_{j_s}\}$ be a subset of X. A *minor* of I is a proper ideal I' of $R' = K[X \setminus X']$ obtained from I by making $x_{i_k} = 0$ and $x_{j_\ell} = 1$ for all k, ℓ. The ideal I is considered itself a minor. A *minor* of $\mathcal{C}$ is a clutter $\mathcal{C}'$ that corresponds to a minor I'.

Recall that a ring is called *reduced* if 0 is its only nilpotent element. The *associated graded ring* of I is the quotient ring $\mathrm{gr}_I(R) := R[It]/IR[It]$.

Corollary 3.9 *If the associated graded ring* $\mathrm{gr}_I(R)$ *is reduced, then* $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ *for any minor* $\mathcal{C}'$ *of* $\mathcal{C}$.

Proof. As the reducedness of $\mathrm{gr}_I(R)$ is preserved if we make a variable x_i equal to 0 or 1, we may assume that $\mathcal{C}' = \mathcal{C}$. From [8, Proposition 3.4] and Theorem 3.2 it follows that the ring $\mathrm{gr}_I(R)$ is reduced if and only if $R[It]$ is normal and $Q(A)$ is integral. Hence by Theorem 3.4 we obtain that the LP-duality equation

$$\min\{\langle \mathbf{1}, x\rangle \mid x \geq 0; xA \geq \mathbf{1}\} = \max\{\langle y, \mathbf{1}\rangle \mid y \geq 0; Ay \leq \mathbf{1}\}$$

has optimum integral solutions x, y. To complete the proof notice that the left hand side of this equality is $\alpha_0(\mathcal{C})$ and the right hand side is $\beta_1(\mathcal{C})$. □

Next we state an algebraic version of a conjecture [6, Conjecture 1.6] which to our best knowledge is still open:

Conjecture 3.10 If $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for all minors $\mathcal{C}'$ of $\mathcal{C}$, then the associated graded ring $\mathrm{gr}_I(R)$ is reduced.

Proposition 3.11 *Let B be the matrix with column vectors $(v_1, 1), \ldots, (v_q, 1)$. If $x^{v_1}, \ldots, x^{v_q}$ are monomials of the same degree $d \geq 2$ and $\mathrm{gr}_I(R)$ is reduced, then B diagonalizes over $\mathbb{Z}$ to an identity matrix.*

Proof. As $R[It]$ is normal, the result follows from [7, Theorem 3.9]. □

This result suggest the following weaker conjecture:

Conjecture 3.12 (Villarreal) Let A be a $\{0, 1\}$-matrix such that the number of 1's in every column of A has a constant value $d \geq 2$. If $\alpha_0(\mathcal{C}') = \beta_1(\mathcal{C}')$ for all minors $\mathcal{C}'$ of $\mathcal{C}$, then the quotient group $\mathbb{Z}^{n+1}/((v_1, 1), \ldots, (v_q, 1))$ is torsion-free.

Symbolic Rees algebras Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of the edge ideal $I = I(\mathcal{C})$ and let $C_k = \{x_i \mid x_i \in \mathfrak{p}_k\}$, for $k = 1, \ldots, s$, be the corresponding minimal vertex covers of the clutter $\mathcal{C}$. We set

$$\ell_k = (\textstyle\sum_{x_i \in C_k} e_i, -1) \quad (k = 1, \ldots, s).$$

The *symbolic Rees algebra* of I is the K-subalgebra:

$$R_s(I) = R + I^{(1)}t + I^{(2)}t^2 + \cdots + I^{(i)}t^i + \cdots \subset R[t],$$

where $I^{(i)} = \mathfrak{p}_1^i \cap \cdots \cap \mathfrak{p}_s^i$ is the ith symbolic power of I.

Corollary 3.13 *The following conditions are equivalent*

(a) $Q(A)$ *is integral.*

(b) $\mathbb{R}_+\mathcal{A}' = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap \cdots \cap H_{\ell_s}^+$.

(c) $\overline{R[It]} = R_s(I)$, *i.e.*, $\overline{I^i} = I^{(i)}$ *for all* $i \geq 1$.

Proof. The integral vertices of $Q(A)$ are precisely the vectors $a_1, \ldots, a_s$, where $a_k = \sum_{x_i \in C_k} e_i$ for $k = 1, \ldots, s$. Hence by Theorem 3.2 we obtain that (a) is equivalent to (b). By [8, Corollary 3.8] we get that (b) is equivalent to (c). □

Corollary 3.14 *Let $\mathcal{C}$ be a clutter and let I be its edge ideal. Then $\mathcal{C}$ has the max-flow min-cut property if and only if* $I^i = I^{(i)}$ *for all* $i \geq 1$.

Proof. It follows at once from Corollary 3.13 and Theorem 3.4. □

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