Blowup algebras of ideals of vertex covers of bipartite graphs

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Abstract

Let G be a bipartite graph. We show some algebraic properties of blowup algebras associated to the ideal of vertex covers of G.

1 Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let I be a monomial ideal of R of height at least two minimally generated by a set of square-free monomials $\{x^{v_1}, \ldots, x^{v_q}\}$ of degree $d \geq 2$. As usual we use x^a as an abbreviation for $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. We associate to the ideal I a *hypergraph* G by taking the set of indeterminates $X = \{x_1, \ldots, x_n\}$ as vertex set and $E = \{S_1, \ldots, S_q\}$ as edge set, where

$$S_k = \{x_i | \langle e_i, v_k \rangle = 1\}.$$

Here $\langle\ ,\ \rangle$ denotes the standard inner product and e_i is the *ith* unit vector. For this reason I is called the *edge ideal* of G. To stress the relationship between I and G we will use the notation $I = I(G)$.

A set $C \subset X$ is a *minimal vertex cover* if every edge of G contains at least one vertex in C and C is minimal with respect to this property. The following are equivalent:

- $\mathfrak{p} = (x_1, \ldots, x_r)$ is a minimal prime of $I = I(G)$.
- $C = \{x_1, \ldots, x_r\}$ is a minimal vertex cover of G.

[0]2000 *Mathematics Subject Classification.* Primary 13H10; Secondary 13F20, 13B22, 52B20.
[1]The authors were partially supported by CONACyT grant 45256

This follows readily by noticing that the minimal primes of I are face ideals, see [14, Proposition 5.1.3]. Recall that $I = I(G)$ is said to be *unmixed* if all the minimal vertex covers of G have the same cardinality. We denote the smallest number of vertices in any minimal vertex cover by $\alpha_0(G)$. Thus $\alpha_0(G) = \mathrm{ht}(I)$, the height of the ideal I.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of $I(G)$ and let

$$C_k = \{x_i \mid x_i \in \mathfrak{p}_k\} \quad (k = 1, \ldots, s)$$

be the corresponding minimal vertex covers of G. We are ready to introduce the principal object of study here:

Definition 1.1 Let $x^{u_k} = \prod_{x_i \in C_k} x_i$ for $1 \leq k \leq s$. The ideal of *vertex covers* of G is the square-free monomial ideal

$$I_c(G) = (x^{u_1}, \ldots, x^{u_s}) \subset R,$$

which in our case is an unmixed ideal of height d.

In the literature $I_c(G)$ is also called the *Alexander dual* of I because if Δ is the Stanley-Reisner complex of I, then $I_c(G)$ is the Stanley-Reisner ideal of Δ^*, the Alexander-dual of Δ. The survey article [5] explains the role of Alexander duality to prove combinatorial and algebraic theorems. This paper is devoted to show some algebraic properties of blowup algebras associated to the ideal of vertex covers of a bipartite graph G.

The *blowup algebras* studied here are the *Rees algebra*

$$R[It] = R \oplus It \oplus \cdots \oplus I^i t^i \oplus \cdots \subset R[t],$$

where t is a new variable, and the *associated graded ring*

$$\mathrm{gr}_I(R) = R[It]/IR[It].$$

If G is a bipartite graph and $J = I_c(G)$ we show that $\mathrm{gr}_J(R)$ is reduced, this complements the fact that $\mathrm{gr}_I(R)$ is reduced [11]. Then we prove that $R[Jt]$ is a Gorenstein ring if I is unmixed and compute its a-invariant. A formula for the a-invariant of $R[It]$ can be found in [4]. For unexplained terminology and notation we refer to [9, 10, 12].

2 Blowup algebras of ideals of covers

The set of non-negative real numbers will be denoted by $\mathbb{R}_+$. If α is a vector we use the notation $\mathbb{R}_+\alpha = \{x\alpha \mid x \in \mathbb{R}_+\}$. Let $\mathcal{A} = \{v_1, \ldots, v_q\}$ be the set of exponent vectors of $x^{v_1}, \ldots, x^{v_q}$ and let

$$\mathcal{A}' = \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_q, 1)\},$$

where e_i is the *ith* unit vector. The *Rees cone* is the rational polyhedral cone, denoted by $\mathbb{R}_+\mathcal{A}'$, consisting of the linear combinations of $\mathcal{A}'$ with non-negative coefficients. Note $\dim(\mathbb{R}_+\mathcal{A}') = n+1$. Thus according to [15] there is a unique irreducible representation

$$\mathbb{R}_+\mathcal{A}' = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap \cdots \cap H_{\ell_r}^+ \tag{1}$$

such that $0 \neq \ell_i \in \mathbb{Q}^{n+1}$ and $\langle \ell_i, e_{n+1}\rangle = -1$ for all i. As usual H_a^+ denotes the closed halfspace

$$H_a^+ = \{\alpha \in \mathbb{R}^{n+1} \,|\, \langle \alpha, a\rangle \geq 0\}$$

and H_a is the hyperplane with normal vector a. According to [2, Lemma 3.1] the irreducible representation of the Rees cone can be written as

$$\mathbb{R}_+\mathcal{A}' = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap \cdots \cap H_{\ell_r}^+, \tag{2}$$

where $\ell_k = -e_{n+1} + \sum_{x_i \in C_k} e_i$ for $k \leq s$. Recall that the *Simis cone* is the rational polyhedral cone

$$\mathrm{Cn}(\mathcal{A}) = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap \cdots \cap H_{\ell_s}^+, \tag{3}$$

and the *symbolic Rees algebra* of I is the finitely generated K algebra:

$$R_s(I) = R + I^{(1)}t + I^{(2)}t^2 + \cdots + I^{(i)}t^i + \cdots \subset R[t],$$

where $I^{(i)} = \mathfrak{p}_1^i \cap \cdots \cap \mathfrak{p}_s^i$ is the *ith* symbolic power of I. The next result highlights the importance of the Simis cone.

Theorem 2.1 ([2]) *If $S = \mathbb{Z}^{n+1} \cap \mathrm{Cn}(\mathcal{A})$ and $K[S] = K[\{x^a t^b |\, (a,b) \in S\}]$ is its semigroup ring, then $R_s(I) = K[S]$.*

Let $\mathbb{N}\mathcal{A}'$ be the subsemigroup of $\mathbb{N}^{n+1}$ generated by $\mathcal{A}'$, consisting of the linear combinations of $\mathcal{A}'$ with non-negative integral coefficients. The Rees algebra can be written as

$$R[It] = K[\{x_1, \ldots, x_n, x^{v_1}t, \ldots, x^{v_q}t\}] = K[\{x^a t^b |\, (a,b) \in \mathbb{N}\mathcal{A}'\}]. \tag{4}$$

According to [14, Theorem 7.2.28] and [13, p. 168] the integral closure of $R[It]$ in its field of fractions can be expressed as

$$\begin{aligned}\overline{R[It]} &= K[\{x^a t^b |\, (a,b) \in \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'\}] \\ &= R \oplus \overline{I}t \oplus \cdots \oplus \overline{I^i}t^i \oplus \cdots, \end{aligned} \tag{5}$$

where $\overline{I^i}$ is the integral closure of I^i. Hence $R[It]$ is a normal domain if and only if any of the following two equivalent conditions hold:

(a) $\mathbb{N}\mathcal{A}' = \mathbb{Z}^{n+1} \cap \mathbb{R}_+\mathcal{A}'$.

(b) $I^i = \overline{I^i}$ for all $i \geq 1$.

Proposition 2.2 *The irreducible representation of the Rees cone has the form*

$$\mathbb{R}_+\mathcal{A}' = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap \cdots \cap H_{\ell_s}^+$$

if and only if $R_s(I) = \overline{R[It]}$.

Proof. It follows from Eq. (2) and Theorem 2.1. $\Box$

Proposition 2.3 *If* $\overline{R[It]} = R_s(I)$ *and* $J = I_c(G)$, *then* $\overline{R[Jt]} = R_s(J)$.

Proof. Using Proposition 2.2 one has

$$\mathbb{R}_+\mathcal{A}' = H_{e_1}^+ \cap \cdots \cap H_{e_{n+1}}^+ \cap H_{\ell_1}^+ \cap \cdots \cap H_{\ell_s}^+,$$

where $\ell_k = (u_k, -1)$ and $x^{u_k} = \prod_{x_i \in C_k} x_i$ for all k. Using the duality theorem for polyhedral cones (see [10, Corollary 7.1a] and its proof) we get:

$$\left(\bigcap_{i=1}^{n} H_{e_i}^+\right) \bigcap \left(\bigcap_{i=1}^{q} H_{(v_i,1)}^+\right) = \left(\sum_{i=1}^{n+1} \mathbb{R}_+ e_i\right) + \left(\sum_{i=1}^{s} \mathbb{R}_+(u_i, -1)\right). \qquad (*)$$

By Proposition 2.2 we need only show the equality

$$\left(\sum_{i=1}^{n} \mathbb{R}_+ e_i\right) + \left(\sum_{i=1}^{s} \mathbb{R}_+(u_i, 1)\right) = \left(\bigcap_{i=1}^{n+1} H_{e_i}^+\right) \bigcap \left(\bigcap_{i=1}^{q} H_{(v_i,-1)}^+\right).$$

Clearly the left hand side is contained in the right hand side. To show the reverse containment take (α, b) in the right hand side, where $\alpha \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Then

$$0 \leq \langle(\alpha, b), (v_i, -1)\rangle = \langle(\alpha, -b), (v_i, 1)\rangle$$

for all i. Hence using Eq. $(*)$ we can write

$$(\alpha, -b) = \sum_{i=1}^{s} \lambda_i(u_i, -1) + \sum_{i=1}^{n+1} \mu_i e_i \quad (\lambda_i; \mu_j \geq 0). \qquad (**)$$

Since $\lambda_1 + \cdots + \lambda_s - b = \mu_{n+1} \geq 0$, there are $\lambda_1', \ldots, \lambda_s'$ such that

$$0 \leq \lambda_i' \leq \lambda_i;\ b = \lambda_1' + \cdots + \lambda_s';\ \mu_{n+1} = (\lambda_1 - \lambda_1') + \cdots + (\lambda_s - \lambda_s').$$

Therefore using Eq. $(**)$ we conclude

$$(\alpha, b) = \sum_{i=1}^{s} \lambda_i'(u_i, 1) + \sum_{i=1}^{n} \mu_i e_i + \sum_{i=1}^{s} (\lambda_i - \lambda_i') u_i,$$

as desired. $\Box$

Theorem 2.4 *Let G be a graph and let $J = I_c(G)$ be its ideal of vertex covers. If $\overline{R[It]}$ is equal to $R_s(I)$, then $R[It]$ and $R[Jt]$ are normal.*

Proof. First we show that the Rees algebra of I is normal by induction on n. Let $x^at^b = x_1^{a_1}\cdots x_n^{a_n}t^b \in \overline{R[It]}$ be a minimal generator, that is (a,b) cannot be written as a sum of two non-zero integral vectors in the Rees cone $\mathbb{R}_+\mathcal{A}'$. By the induction procedure we may assume $a_i \geq 1$ for all i and $b \geq 1$.

Case (I): $\langle (a,b),\ell_i\rangle > 0$ for all i. The vector $\gamma = (a,b) - e_1$ satisfies $\langle \gamma,\ell_i\rangle \geq 0$ for all i, that is $\gamma \in \mathbb{R}_+\mathcal{A}'$. Thus since $(a,b) = e_1 + \gamma$ we derive a contradiction.

Case (II): $\langle (a,b),\ell_i\rangle = 0$ for some i. We may assume

$$\{\ell_i |\, \langle (a,b),\ell_i\rangle = 0\} = \{\ell_1,\ldots,\ell_p\}.$$

Subcase (II.a): $e_i \in H_{\ell_1}\cap\cdots\cap H_{\ell_p}$ for some $1 \leq i \leq n$. It is not hard to verify that the vector $\gamma = (a,b) - e_i$ satisfies $\langle \gamma,\ell_k\rangle \geq 0$ for all $1 \leq k \leq s$. Thus $\gamma \in \mathbb{R}_+\mathcal{A}'$, a contradiction because $(a,b) = e_i + \gamma$.

Subcase (II.b): $e_i \notin H_{\ell_1}\cap\cdots\cap H_{\ell_p}$ for all $1 \leq i \leq n$. The vector (a,b) belongs to the polyhedral cone

$$C = H_{\ell_1}\cap\cdots\cap H_{\ell_p}\cap \mathbb{R}_+\mathcal{A}'.$$

Hence we can write

$$\begin{aligned}
(a,b) &= \lambda_1(v_1,1)+\cdots+\lambda_q(v_q,1)+\mu_1e_1+\cdots+\mu_ne_n \qquad (\lambda_i;\mu_j \geq 0),\\
\langle (a,b),\ell_k\rangle &= \lambda_1\langle (v_1,1),\ell_k\rangle+\cdots+\lambda_q\langle (v_q,1),\ell_k\rangle +\\
&\quad\ \mu_1\langle e_1,\ell_k\rangle+\cdots+\mu_n\langle e_n,\ell_k\rangle = 0
\end{aligned}$$

for $k = 1,\ldots,p$. If $\mu_i > 0$ for some $1 \leq i \leq n$, then $\langle e_i,\ell_k\rangle = 0$ for $1 \leq k \leq p$, a contradiction. Hence $\mu_i = 0$ for all i. Therefore

$$(a,b) = \lambda_1(v_1,1)+\cdots+\lambda_q(v_q,1). \tag{$*$}$$

By the choice of x^at^b we may assume $0 < \lambda_1 < 1$. Set $\gamma = (a,b) - (v_1,1)$ and notice that by Eq. $(*)$ this vector has non-negative entries. We claim that γ is in the Rees cone. Since one has $0 \leq \langle (v_1,1),\ell_j\rangle \leq 1$ for all j we readily obtain

$$\langle \gamma,\ell_k\rangle = \left\{\begin{array}{lcll} \langle (a,b),\ell_k\rangle - \langle (v_1,1),\ell_k\rangle & = & 0 & \text{if } 1 \leq k \leq p,\\ \langle (a,b),\ell_k\rangle - \langle (v_1,1),\ell_k\rangle & \geq & 0 & \text{otherwise.}\end{array}\right.$$

Thus $\gamma \in \mathbb{R}_+\mathcal{A}'$ and $(a,b) = (v_1,1)+\gamma$. As a result $\gamma = 0$ and $(a,b) \in R[It]$, as desired. To show the normality of $R[Jt]$ it suffices to note that by Proposition 2.3 $\overline{R[Jt]} = R_s(J)$ and to apply the arguments above. $\Box$

A ring is called *reduced* if 0 is its only nilpotent element. For convenience let us state some known characterizations of the reducedness of the associated graded ring.

Theorem 2.5 ([2, 8]) *The following conditions are equivalent*

(i) $\mathrm{gr}_I(R)$ *is reduced.*

(ii) $R[It]$ *is normal and* $s=r$.

(iii) I *is normally torsion free, that is* $I^i = I^{(i)}$ *for all* $i \geq 1$.

Corollary 2.6 *If* G *is a bipartite graph and* $J = I_c(G)$, *then* $\mathrm{gr}_J(R)$ *is reduced.*

Proof. According to [11, Theorem 5.9] one has the equality $R[It] = R_s(I)$. Thus Proposition 2.3 yields the equality $\overline{R[Jt]} = R_s(J)$. On the other hand $R[Jt]$ is normal by Theorem 2.4. Thus by Theorem 2.5 the associated graded ring of J is reduced. $\square$

By assigning $\deg(x_i) = 1$ and $\deg(t) = -(d-1)$, the Rees algebra $R[It]$ becomes a standard graded K-algebra generated by elements of degree 1. Below we compute the a-invariant of $R[It]$ with respect to this grading.

Theorem 2.7 *If* $\mathrm{gr}_I(R)$ *is a reduced ring and* $I = I(G)$ *is an unmixed ideal of height* 2, *then* $R[It]$ *is Gorenstein and*

$$a(R[It]) = -(n-d+1).$$

Proof. Since $R[It]$ is a normal domain, then according to a formula of Danilov-Stanley [1, Theorem 6.3.5] its *canonical module* is the ideal of $R[It]$ given by

$$\omega_{R[It]} = (\{x_1^{a_1}\cdots x_n^{a_n}t^{a_{n+1}} \,|\, a=(a_i) \in (\mathbb{R}_+\mathcal{A}')^{\mathrm{o}} \cap \mathbb{Z}^{n+1}\}),$$

where $(\mathbb{R}_+\mathcal{A}')^{\mathrm{o}}$ is the topological interior of the Rees cone. By a result of M. Hochster [7] the ring $R[It]$ is Cohen-Macaulay. In our situation it is well known (see [1]) that the a-invariant can be expressed as

$$a(R[It]) = -\min\{\, i \,|\, (\omega_{R[It]})_i \neq 0\}.$$

Using Eq. (2) it is seen that the vector $(1,\ldots,1,1)$ is in the interior of the Rees cone. Thus $a(R[It])$ is at least $-(n-d+1) = -\deg(x_1\cdots x_n t)$. To show the reverse inequality take an arbitrary monomial $x^at^b = x_1^{a_1}\cdots x_n^{a_n}t^b$ in the ideal $\omega_{R[It]}$, that is $(a,b) \in (\mathbb{R}_+\mathcal{A}')^{\mathrm{o}}$. By Proposition 2.2 the vector (a,b) has positive entries and satisfies

$$-b + \textstyle\sum_{x_i\in C_k} a_i \geq 1 \quad (k=1,\ldots,s). \tag{6}$$

If $b=1$ we obtain the inequality

$$\deg(x^at^b) = a_1+\cdots+a_n-(d-1) \geq n-d+1. \tag{$*$}$$

Now assume $b \geq 2$. Using the normality of $R[It]$ and Eqs. (2) and (6) it follows that the monomial

$$m = x_1^{a_1-1} \cdots x_n^{a_n-1} t^{b-1}$$

belongs to $R[It]$. Since $x^a t^b = m x_1 \cdots x_n t$, the inequality $(*)$ also holds in this case. Altogether we conclude the desired equality for the a-invariant. From the argument above we obtain that $\omega_{R[It]}$ is generated by $x_1 \cdots x_n t$ and thus $R[It]$ is a Gorenstein ring. $\Box$

Corollary 2.8 *Let $J = I_c(G)$ be the ideal of vertex covers of G. If G is a bipartite graph and $I = I(G)$ is unmixed, then $R[Jt]$ is a Gorenstein ring and*

$$a(R[Jt]) = -(n - \alpha_0(G) + 1).$$

Proof. It follows from Theorem 2.7 once we recall that $\mathrm{gr}_J(R)$ is a reduced ring according to Corollary 2.6. $\Box$

The family of unmixed bipartite graphs contains the class of Cohen-Macaulay bipartite graphs [14, Proposition 6.1.21]. The latter class is constructible [3] and has been classified using commutative algebra and combinatorial techniques [6].

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